

Nonlinear effects in charged particle transport in turbulent magnetic fields

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Abstract. The problem of charged particle transport in stochastic magnetic fields is analyzed using a new statistical approaches, the decorrelation trajectory method and the nested subensemble method. The Lagrangian non-linearity determined by the space-dependence of the stochastic magnetic field produces a process of magnetic line trapping, which appears at large magnetic Kubo numbers. This trapping process leads to segments of the magnetic lines with helicoidal shapes, which form localized stochastic structures similar to magnetic islands. They strongly influence the particle transport in stochastic magnetic fields. Two applications are considered in order to analyze the nonlinear effects of the stochastic magnetic islands: collisional particle transport and the diffusion of the energetic particles with large Larmor radius.

Keywords. space plasma, turbulent transport, stochastic magnetic field

1. Introduction

The transport of charge particles in stochastic magnetic field is an important problem in many astrophysical issues as cosmic rays in heliosphere, energetic particle propagation in the solar wind, Fermi acceleration process, galactic cosmic rays, etc. The problem of test particle diffusion in stochastic magnetic fields is an active topic studied by many authors (see for instance Vlad et al. 2003, Pommois et al. 2007, Shalchi et al. 2004, McKibben 2005 and the references therein). Important progress was obtained since the first papers of Jakipii & Parker 1969 and Rechester & Rosenbluth 1978. However, the general solution has not yet been found. The main difficulty is related with the Lagrangian non-linearity which is determined by the space dependence of the stochastic magnetic field. Most of the studies are based on the quasilinear theory of stochastic transport, on numerical simulations or on phenomenological models. The nonlinearity can determine a strong influence on the transport coefficient and on the statistical characteristics of the trajectories. These nonlinear processes were recently studied in the context of laboratory magnetized turbulent plasmas (fusion experiments). The transport induced by the $\mathbf{E} \times \mathbf{B}$ stochastic drift in electrostatic turbulence (including effects of collisions, average flows, motion along magnetic field), the transport in magnetic turbulence and the Lorentz transport for arbitrary Larmor radius and cyclotron frequency were studied in a series of papers (Vlad et al. 2000, Vlad et al. 2001, Vlad et al. 2002, Vlad et al. 2004). The results are rather unexpected when the nonlinear effects are strong. The diffusion coefficients are completely different of those obtained in quasilinear conditions and the dependence on the specific parameters is reversed. Moreover the statistics of trajectories is non-standard, with non-Gaussian distribution, memory effects and high degree of coherence. These studies are based on new statistical approaches: the decorrelation trajectory method [Vlad et al. 1998] and the nested subensemble method [Vlad & Spineanu 2004].

The aim of this paper is to present these semi-analytical statistical methods and some

results on nonlinear effects appearing in charged particle transport in magnetic turbulence. Most of these studies were done for the parameters of fusion plasmas that are completely different of those of space plasmas. However, since the transport process depends on dimensionless numbers, some of the results are directly relevant for space plasmas. Certainly, the methods we have developed can be extended to specific particle transport problems in space plasmas. We present in Sec. 2 the process of diffusion by continuous movements and emphasize the special case of 2-dimensional stochastic velocity fields where trapping or eddying motion can appear. The statistical methods that are able to describe the effects of trajectory trapping are described in Sec. 3. The next three sections present applications: diffusion of magnetic lines and magnetic structure formation (Sec. 4), particle diffusion in stochastic magnetic fields (Sec. 5) and the effect of Larmor radius on the transport coefficients (Sec. 6). The conclusions are summarized in Sec. 7.

2. Diffusion by continuous movements

The problem of stochastic advection or diffusion by continuous movements is described by stochastic equations of the type:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}[\mathbf{x}(t), t], \quad \mathbf{x}(0) = \mathbf{0} \quad (2.1)$$

where $\mathbf{x}(t)$ represents the trajectory. The velocity $\mathbf{v}(\mathbf{x}, t)$ is statistically described. It is considered to be a stationary and homogeneous Gaussian stochastic field, with zero average and given two-point Eulerian correlation function (EC)

$$E_{ij}(\mathbf{x}, t) \equiv \langle v_i(\mathbf{x}_1, t_1) v_j(\mathbf{x}_1 + \mathbf{x}, t_1 + t) \rangle \quad (2.2)$$

where $\langle \dots \rangle$ denotes the statistical average over the realizations of $\mathbf{v}(\mathbf{x}, t)$. The velocity is a continuous function of \mathbf{x} and t in each realization and it determines an unique trajectory as the solution Eq. (2.1). Starting from the above statistical description of the stochastic velocity and from an explicit EC, $E(\mathbf{x}, t)$, one has to determine the statistical properties of the trajectories. This problem is nonlinear due to the space dependence of the velocity, which leads to \mathbf{x} -dependence of the EC (2.2). The importance of the nonlinearity is characterized by the Kubo number defined by

$$K = \frac{V\tau_c}{\lambda_c} = \frac{\tau_c}{\tau_{\text{fl}}} \quad (2.3)$$

where V is the amplitude of the stochastic velocity, τ_c is the correlation time and λ_c is the correlation length. These parameters appear in the EC of the velocity as the maximum value in the origin [$V^2 = E_{ii}(\mathbf{0}, 0)$] and the characteristic decay time and length of this functions. The Kubo number is thus the ratio of τ_c to the average time of flight of the particles over the correlation length, $\tau_{\text{fl}} = \lambda_c/V$. It measures the particle's capacity of exploring the space structure of the stochastic velocity field before it changes.

As shown by Taylor 1921, the mean square displacement $\langle x_i^2(t) \rangle$ and its derivative, the running diffusion coefficient $D_i(t)$, are determined by the Lagrangian velocity correlation (LVC), defined by

$$L_{ij}(t) \equiv \langle v_i[\mathbf{x}(0), 0] v_j[\mathbf{x}(t), t] \rangle \quad (2.4)$$

for a stationary process, as:

$$\langle x_i^2(t) \rangle = 2 \int_0^t d\tau L_{ii}(\tau) (t - \tau), \quad (2.5)$$

$$D_i(t) = \int_0^t d\tau L_{ii}(\tau). \quad (2.6)$$

The diffusion coefficient is the asymptotic value of $D(t)$.

Several methods (as Corrsin approximation, Corrsin 1959, and direct interaction approximation, Roberts 1961) were developed for determining the LVC corresponding to given EC. See also McComb 1990 and Krommes 2002. They are based on the hypothesis of Gaussian statistics of trajectories. Actually, the scaling of the diffusion coefficient in the Kubo number can be obtained by simple estimation based on the general shape of the LVC (2.4). It is usually a function that decays to zero from the value V^2 in $t = 0$. For small Kubo numbers $K \ll 1$ ($\tau_c \ll \tau_{\text{fl}}$), the time variation of the velocity field is fast and the particles cannot “see” the space structure of the velocity field. In this quasilinear regime (or the weak turbulence case) the LVC decays in a time of the order of τ_c and the diffusion coefficient obtained from (2.6) is $D_{ql} \approx V^2 \tau_c = (\lambda_c^2 / \tau_c) K^2$. In the nonlinear regime $K \gg 1$ ($\tau_{\text{fl}} \ll \tau_c$), the decay time of the LVC is τ_{fl} and the diffusion scales as $D \approx V^2 \tau_{\text{fl}} = (\lambda_c^2 / \tau_c) K$. The shape of the EC determines only a numerical factor in the diffusion coefficient.

The two dimensional velocity fields $\mathbf{v}(\mathbf{x}, t)$ that are divergence-free, $\nabla \cdot \mathbf{v}(\mathbf{x}, t) = 0$, represent a special case that cannot be describe by the above methods. The two components \mathbf{v} can be determined from a stochastic scalar field, the potential $\phi(\mathbf{x}, t)$, as:

$$v_i(\mathbf{x}, t) = \varepsilon_{ij} \frac{\partial \phi(\mathbf{x}, t)}{\partial x_j} \quad (2.7)$$

where ε_{ij} is the antisymmetric tensor ($\varepsilon_{12} = 1$, $\varepsilon_{21} = -1$, $\varepsilon_{11} = \varepsilon_{22} = 0$). The trajectories are solution of a Hamiltonian system and thus for time independent potential they are closed curves that lie on contour lines of $\phi(\mathbf{x})$. This invariance property is approximately maintained for potentials with slow time variation characterized by $K > 1$ ($\tau_c > \tau_{\text{fl}}$). The trajectories have segments that approximately follow the contour lines of $\phi(\mathbf{x}, t)$ for a time that is larger than τ_{fl} . This produces a trapping effect : the trajectories are confined for long periods in small regions. A typical trajectory shows an alternation of large displacements and trapping events. The latter appear when the particles are close to the maxima or minima of the potential and consist of trajectory winding on almost closed paths. The large displacements are produced when the trajectories are at small absolute values of the potential. Trajectory trapping appears for $K > 1$ and becomes stronger as K increases up to the limit of static fields ($K, \tau_c = \infty$) where the trapping is permanent.

This special problem of diffusion in 2-dimensional divergence-free velocity fields describes for instance the transport in magnetic or electrostatic turbulence in magnetized plasmas or in incompressible fluids. It was studied especially by means of direct numerical simulations (Reuss & Misguich 1996 and the reference there in) or on the basis of simplified models (Majda & Kramer 1999, Pécseli & Trulsen 1997). The analytical methods (Corrsin and direct interaction approximations) are not adequate because the trajectories have non-Gaussian statistics.

Important analytical results in the study of this special case were obtained in the last years. New statistical methods were developed (Vlad et al. 1998, Vlad & Spineanu 2004) that permitted to determine the statistical characteristics of the trajectories. This method could be extended to more complicated physical systems which contain particle collisions, average velocities or a supplementary component of the motion perpendicular to the 2-dimensional plane (Vlad et al. 2002 and the references there in). It was shown that the presence of trapping determines memory effects in $L(t)$ and a rich class of anomalous

diffusion regimes in the presence of a decorrelation mechanism Vlad & Spineanu 2003. The trapping has also collective effects. It determines coherence in the stochastic motion in the sense that bundles of neighboring trajectories form localized structures similar with fluid vortices. We present below the statistical methods and the main characteristics of the stochastic trajectories.

3. Statistical methods

Trajectory trapping is essentially related to the invariance of the Lagrangian potential. Thus, a statistical method is adequate for the study of this process if it is compatible with the invariance of the potential. Such methods are presented in Vlad et al. 1998, Vlad & Spineanu 2004. The main idea in our approach is to study the stochastic equation (2.1) in subensembles of realizations of the stochastic field. First the whole set of realizations R is separated in subensembles ($S1$), which contain all realizations with given values of the potential and of the velocity in the starting point of the trajectories $\mathbf{x} = \mathbf{0}$, $t = 0$:

$$(S1): \quad \phi(\mathbf{0}, 0) = \phi^0, \quad \mathbf{v}(\mathbf{0}, 0) = \mathbf{v}^0. \quad (3.1)$$

Then, each subensemble ($S1$) is separated in subensembles ($S2$) corresponding to fixed values of the second derivatives of the potential in $\mathbf{x} = \mathbf{0}$, $t = 0$

$$(S2): \quad \phi_{ij}(\mathbf{0}, 0) \equiv \left. \frac{\partial^2 \phi(\mathbf{x}, t)}{\partial x_i \partial x_j} \right|_{\mathbf{x}=\mathbf{0}, t=0} = \phi_{ij}^0 \quad (3.2)$$

where $ij = 11, 12, 22$. Continuing this procedure up to an order n , a system of nested subensembles is constructed. The stochastic (Eulerian) potential and velocity in a subensemble are Gaussian fields but non-stationary and non-homogeneous, with space and time dependent averages and correlations. The correlations are zero in $\mathbf{x} = \mathbf{0}$, $t = 0$ and increase with the distance and time. The average potential and velocity performed in a subensemble depend on the parameters of that subensemble and of the subensembles that include it. They are determined by the Eulerian correlation of the potential (see Vlad & Spineanu 2004 for details). The stochastic equation (2.1) is studied in each highest order subensemble (Sn). The average Eulerian velocity determines an average motion in each (Sn). Neglecting the fluctuations of the trajectories, the average trajectory in (Sn), $\mathbf{X}(t; Sn)$, is obtained from

$$\frac{d\mathbf{X}(t; Sn)}{dt} = \varepsilon_{ij} \frac{\partial \Phi(\mathbf{X}; Sn)}{\partial X_j}. \quad (3.3)$$

This approximation consists in neglecting the fluctuations of the trajectories in the subensemble (Sn). It is rather good because it is performed in the subensemble (Sn) where the trajectories are similar due to the fact that they are super-determined. Besides the necessary and sufficient initial condition $\mathbf{x}(0) = \mathbf{0}$, they have supplementary initial conditions determined by the definition (3.1)–(3.2) of the subensembles. The strongest condition is the initial potential $\phi(\mathbf{0}, 0) = \phi^0$ that is a conserved quantity in the static case and determines comparable sizes of the trajectories in a subensemble. Moreover, the amplitude of the velocity fluctuations in (Sn), the source of the trajectory fluctuations, is zero in the starting point of the trajectories and reaches the value corresponding to the whole set of realizations only asymptotically. This reduces the differences between the trajectories in (Sn) and thus their fluctuations.

The statistics of trajectories for the whole set of realizations (in particular the LVC) is obtained as weighted averages of these trajectories $\mathbf{X}(t; Sn)$. The weighting factor is

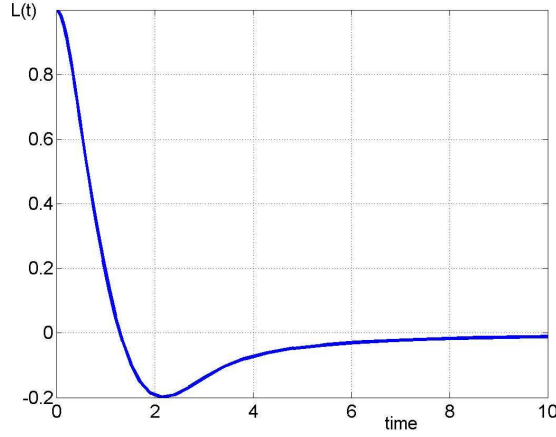


Figure 1. The Lagrangian velocity correlation for static potential

the probability that a realization belongs to the subensemble (S_n); it is analytically determined.

Essentially, this method reduces the problem of determining the statistical behavior of the stochastic trajectories to the calculation of weighted averages of some smooth, deterministic trajectories determined from the EC of the stochastic potential. This semi-analytical statistical approach (the nested subensemble method) is a systematic expansion that satisfies at each order $n > 1$ all statistical conditions required by the invariance of the Lagrangian potential in the static case. At order $n = 1$ only the average potential is conserved. This method is quickly convergent. This is a consequence of the fact that the mixing of periodic trajectories, which characterizes this nonlinear stochastic process, is directly described at each order of our approach. The results obtained in first order (the decorrelation trajectory method) for $D(t)$ are practically not modified in the second order Vlad & Spineanu 2004. Thus, the decorrelation trajectory method is a good approximation for determining diffusion coefficients. This method and the main physical results concerning diffusion coefficients are presented in the next section. The second order nested subensemble method is important because it provides detailed statistical information on trajectories: the probability of the displacements and of the distance between neighboring trajectories in the whole ensemble of realizations and also in the subensembles (S_1). A high degree of coherence is so evidenced in the stochastic motion of trapped trajectories. These aspects are discussed in Sec. 3.2.

3.1. Memory effects and anomalous diffusion coefficients

The decorrelation trajectory method is the first order approximation of the nested subensemble approach and consists in neglecting the fluctuations of the trajectories in the subensembles (S_1). Thus, an equation of the type (3.3) is obtained in each subensemble (S_1). The following closed system of equations is obtained for the LVC and $D(t)$:

$$D(t) \equiv D_B F[\theta(t)], \quad (3.4)$$

$$L(t) = V^2 F'[\theta(t)] h(t) \quad (3.5)$$

where

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dp \int_0^\infty du u^3 \exp\left(-\frac{u^2(1+p^2)}{2}\right) X_1(ut, p) \quad (3.6)$$

and $F'(t) \equiv dF/dt$. X_1 is the component of the solution of Eq.(3.3) along x axis, i.e. the average trajectory in (S1) along \mathbf{v}^0 . The stochastic potential is taken isotropic and the parameters of (S1) subensembles are represented by $u = |\mathbf{v}^0|$ and $p = \phi^0/u$ so that $\mathbf{X}(t; S1) \equiv \mathbf{X}(ut, p)$ (see Vlad et al. 1998 for details). The average potential $\Phi(\mathbf{X}, t; S1)$ is obtained from the Eulerian correlation of the potential, $E(\mathbf{x}, t) = \mathcal{E}(\mathbf{x})h(t)$, as

$$\Phi^S(\mathbf{x}, t; S1) \equiv \langle \phi(\mathbf{x}, t) \rangle_{S1} = \left[\phi^0 \frac{\mathcal{E}(\mathbf{x})}{\mathcal{E}(\mathbf{0})} + v_1^0 \frac{\mathcal{E}_{1\phi}(\mathbf{x})}{\mathcal{E}_{11}(\mathbf{0})} + v_2^0 \frac{\mathcal{E}_{2\phi}(\mathbf{x})}{\mathcal{E}_{22}(\mathbf{0})} \right] h(t). \quad (3.7)$$

where $\mathcal{E}_{ij}(\mathbf{x}) = -\varepsilon_{in}\varepsilon_{jm}\partial^2\mathcal{E}(\mathbf{x})/\partial x_n\partial x_m$ and $\mathcal{E}_{i\phi}(\mathbf{x}) = \varepsilon_{in}\partial\mathcal{E}(\mathbf{x})/\partial x_n$. The time is normalized with τ_{fl} and X_1 with λ_c in Eqs. (3.4)–(3.6). The function $\theta(t)$ determines the effect of time dependence of the potential and is defined by

$$\theta(t) = \int_0^t h(\tau)d\tau, \quad (3.8)$$

It is linear at small time $\theta(t) \cong t$ and saturates at the correlation time of the stochastic potential, which in these units is K , $\theta(t) \xrightarrow[t \rightarrow \infty]{} K$. The asymptotic value for $t \rightarrow \infty$ of the diffusion coefficient (3.4) is:

$$D = D_B F(K) \quad (3.9)$$

where $D_B = V\lambda_c$.

The Lagrangian velocity correlation (2.4) is also a measure of the statistical memory of the stochastic motion. The average Lagrangian velocity at time t for the trajectories with initial velocity \mathbf{v}^0 is $V_i(t; \mathbf{v}^0) = v_i^0 L_{ii}(t)/V^2$. At $t = 0$, $V_i(0; \mathbf{v}^0) = v_i^0$ and the memory of the initial velocity is lost in the stochastic motion according to the time decay of the LVC. In the absence of trapping, the typical LVC for a static field is a function that decay to zero in a time of the order τ_{fl} . This leads to asymptotic diffusion coefficients that are linear in K , $D = cV^2\tau_{\text{fl}} = cD_B = c(\lambda_c^2/\tau_c)K$, where only the constant c is influenced by the EC of the stochastic field.

The presence of trajectory trapping leads to a completely different shape of the LVC. A typical example of the LVC obtained for a static potential ($h(t) = 1$) from Eqs. (3.5), (3.6) is presented in Fig. 1. This function decays to zero in a time of the order τ_{fl} but at later times it becomes negative, it reaches a minimum and then it decays to zero having a long, negative tail. The tail has a power law decay with an exponent that depends on the EC of the potential (Vlad et al. 2004). The positive and negative parts compensate such that the integral of $L(t)$, the running diffusion coefficient $D(t)$, decays to zero. The transport in such 2-dimensional potential is thus subdiffusive, i.e., with mean square displacement that grows with time slower than linearly. The long time tail of the LVC shows that the stochastic trajectories in static potential have a long time memory of the initial velocity.

This stochastic process is unstable in the sense that any weak perturbation produces a strong influence on the transport. A perturbation represents a decorrelation mechanism and its strength is characterized by a decorrelation time τ_d . The weak perturbations produce long decorrelation times, $\tau_d \gg \tau_{\text{fl}}$. In the absence of trapping, such a weak perturbation does not produce a modification of the diffusion coefficient because the LVC is zero at $t > \tau_{\text{fl}}$. In the presence of trapping which is characterized by long time LVC as in Fig. 1, such perturbation influences the tail of the LVC and destroys the equilibrium between the positive and the negative parts. Consequently, the diffusion coefficient depends on the type of decorrelation but it is in general a decreasing function of τ_d . It means that when the decorrelation mechanism becomes stronger (τ_d decreases) the transport increases. This is a consequence of the fact that the long time LVC is

negative. This behavior is completely different of that obtained in stochastic fields that do not produce trapping. In this case, the transport is stable to the weak perturbations. An influence of the decorrelation can appear only when the latter is strong such that $\tau_d < \tau_H$ and it determines the decrease of the diffusion coefficient as the perturbation becomes stronger (τ_d decreases).

The decorrelation can be produced for instance by the time variation of the stochastic potential, which destroys the Lagrangian correlations at $t > \tau_c$. The transport becomes diffusive with an asymptotic diffusion coefficient that scales as $D_{tr} = cV\lambda_c/K^\gamma$, with $0 < \gamma < 1$ (trapping scaling), and thus it is a decreasing function of τ_c . This inverse behavior is determined by the fact that a stronger perturbation (with smaller τ_d) liberates a larger number of trajectories, which contribute to the diffusion. Other types of perturbations were studied with the decorrelation trajectory method (particle collisions Vlad et al. 2000, an average component of the velocity Vlad et al. 2001, etc.). Their interaction with the trapping process produces more complicated nonlinear effects that depend on the specific decorrelation mechanism but all of them have a common property: anomalous diffusion regimes, with diffusion coefficient increasing as the decorrelation becomes stronger, always appear when trajectory trapping is effective.

3.2. Coherence and trajectory structures

The nested subensemble method in the second order provides detailed statistical information. It is possible to determine the statistics of the trajectories that start in points with given values of the potential and also the statistics of the distance between two neighbour trajectories. The effective calculations, which follow the main steps presented in Sec. 3.1, are more complicated as well as the equations for the statistical quantities. They can be found in [Vlad & Spineanu 2004] and we review here the main results.

Two types of trajectories, trapped and free trajectories, are evidenced for a stochastic potential with large Kubo number. The trapped trajectories correspond to subensembles (S1) with large values of $|\phi^0|$ while the free trajectories correspond to $\phi^0 \cong 0$. They have completely different statistical behavior. The trapped trajectories have a quasi-coherent behavior. Their average displacement, dispersion and probability distribution function saturate in a time τ_s . The time evolution of the square distance between two trajectories $\langle \delta x^2(t) \rangle$ is very slow showing that neighboring particles have a coherent motion for a long time, much longer than τ_s . They are characterized by a strong clump effect with the increase of the $\langle \delta x^2(t) \rangle$ that is slower than the Richardson law. These trajectories form structures which are similar with fluid vortices and represent eddying regions. The statistical parameters of these structures (size, build-up time, dispersion) are determined. The dispersion of the trajectories in such a structure is of the order of its size. The size and the built-up time depend on the parameters of the subensemble (S1). They increase when $|\phi^0|$ decreases and become infinite at $\phi^0 = 0$. Thus the trajectory structures appear with all sizes, but their characteristic formation time increases with the size. These structures or eddying regions are permanent in static stochastic potentials. The saturation time τ_s represents the average time necessary for the formation of the structure.

In time dependent potentials the structures with $\tau_s > \tau_c$ are destroyed and the corresponding trajectories contribute to the diffusion process. These free trajectories have a continuously growing average displacement and dispersion. They have incoherent behavior and the clump effect is absent. The probability distribution functions for both types of trajectories are non-Gaussian. The average size S of the structures in a time dependent potential is depends on the Kubo number. For $K < 1$ the structures are absent ($S = 0$) and they appear for $K > 1$ and continuously grow as K increases. The dependence on K is a power law with the exponent dependent on the Eulerian correlation of the potential.

4. Magnetic line diffusion

We consider a stochastic magnetic field modelled by

$$\mathbf{B} = B_0 \left(\mathbf{e}_z + \tilde{\mathbf{b}}(\mathbf{x}, z, t) \right) \quad (4.1)$$

where there is an average constant component $\mathbf{B}_0 = B_0 \mathbf{e}_z$ directed along z axis and a fluctuating field perpendicular to \mathbf{B}_0 , and depending on the perpendicular coordinates $\mathbf{x} \equiv (x, y)$ and on the parallel coordinate z . Since the magnetic field is divergence-free, $\nabla \cdot \tilde{\mathbf{b}} = 0$, its two components can be determined from a scalar function $\tilde{\phi}(\mathbf{x}, z)$ as

$$\tilde{\mathbf{b}}(\mathbf{x}, z) = \nabla \times \tilde{\phi}(\mathbf{x}, z) \mathbf{e}_z. \quad (4.2)$$

The system of equations for the magnetic line displacement is:

$$\frac{d\mathbf{x}}{dz} = \tilde{\mathbf{b}}(\mathbf{x}, z, t). \quad (4.3)$$

The magnetic lines are thus described by stochastic equations of the type 2.1, 2.7 with the time variable replaced by the z coordinate. The equivalent Kubo number is here the magnetic Kubo number defined by

$$K_m = \frac{\beta \lambda_{\parallel}}{\lambda_{\perp}} \quad (4.4)$$

where β is the amplitude of the magnetic field fluctuations $\tilde{\mathbf{b}}$, λ_{\parallel} is the correlation length of the potential $\tilde{\phi}$ along the main magnetic field \mathbf{B}_0 and λ_{\perp} is the correlation length in the plane perpendicular to \mathbf{B}_0 . It is a dimensionless measure of the variation of the magnetic field fluctuations along the average component $\mathbf{B}_0 = B_0 \mathbf{e}_z$.

The nonlinear effects in the statistics of the magnetic lines appear when $K_m > 1$ due to trapping. The latter is represented by solenoidal segments of the magnetic lines oriented along \mathbf{B}_0 . As discussed in the previous section, the trapped magnetic lines have quasi-coherent behaviour and form structures that are localized magnetic islands in the stochastic magnetic field. Their length along z axis is of the order of λ_{\parallel} and their perpendicular size increases when the magnetic Kubo number increases covering an increased fraction of the volume. At the limit $\lambda_{\parallel} \rightarrow \infty$ the structure of the magnetic lines is regular in the whole volume, with magnetic islands of all sizes. Due to the existence of such stochastic magnetic islands, the space diffusion of the magnetic lines perpendicular to the average magnetic field \mathbf{B}_0 is strongly reduced and the diffusion coefficient D_m has an anomalous scaling with the parallel correlation length $D_m = \beta \lambda_c / K_m$. Thus it decreases with the increase of λ_{\parallel} .

Particle motion in stochastic magnetic fields is much influenced by the presence of stochastic magnetic islands. This influence is obvious when the particles follow the magnetic lines, i.e., for negligible collisions, Larmor radius ρ much smaller than λ_{\perp} and large average cyclotron frequency $\Omega = qB_0/m \gg \tau_{\text{fl}}^{-1}$. In this conditions, the particle diffusion coefficient is $D = v_{\parallel} D_m$, where v_{\parallel} is the velocity along the average magnetic field. Strong nonlinear effects of the stochastic magnetic islands also appear in the presence of collisions and at large ρ . They are presented in the next sections.

5. Collisional particle diffusion

The transport of collisional particles in the stochastic magnetic field 4.1 in the guiding center approximation is determined from the system of stochastic equations:

$$\frac{d\mathbf{x}}{dt} = \tilde{\mathbf{b}}(\mathbf{x}, z)\eta_{\parallel}(t) + \eta_{\perp}(t), \quad (5.1)$$

$$\frac{dz}{dt} = \eta_{\parallel}(t). \quad (5.2)$$

It is a rather complex process determined by three stochastic functions, which interact due to the space dependence of the stochastic magnetic field. The parallel $\eta_{\parallel}(t)$ and the perpendicular $\eta_{\perp}(t)$ velocities are modelled by colored Gaussian noises with collisional diffusion coefficients $\chi_{\parallel} = \lambda_{\text{mfp}}^2\nu/2$ and $\chi_{\perp} = \rho^2\nu/2$ where λ_{mfp} is the parallel mean free path, ν is the collision frequency and ρ is the Larmor radius determined by the average magnetic field. Three dimensionless parameters appear naturally in this problem: the dimensionless perpendicular and respectively parallel diffusivities $\bar{\chi}_{\perp} \equiv \chi_{\perp}/(\lambda_{\perp}^2\nu)$, $\bar{\chi}_{\parallel} \equiv \chi_{\parallel}/(\lambda_{\parallel}^2\nu)$, and a dimensionless parameter that contains the effect of the stochastic magnetic field $M = K_{\text{m}}\bar{\chi}_{\parallel}^{-1/2}$.

The first semi-analytical study of this complex process was done by Vlad et al. 2003 by developing the decorrelation trajectory method. It was shown that the stochastic magnetic islands can have a very strong effect on particle transport and the transport regimes were determined. The main effects of the stochastic magnetic lines can be understood by considering first particles that have negligible $\bar{\chi}_{\perp}$ and thus follow the magnetic lines. In the quasilinear conditions $K_{\text{m}}, M \ll 1$ the exact solution $D_0(t)$ can be obtained which shows the process determined by the two (multiplicative) stochastic functions $\tilde{\mathbf{b}}(\mathbf{x}, z)$ and $\eta_{\parallel}(t)$ is subdiffusive with $D_0(t) \rightarrow 0$ and the mean square displacement growing as $t^{1/2}$. It is shown that the generation of magnetic islands by magnetic line trapping does not change the asymptotic behavior of the diffusion coefficient but determines a transient decrease of $D(t)$. The nonlinear result is presented in Fig. 3 compared to the quasilinear solution $D_0(t)$. One can see that at small and large times the diffusion coefficient is equal to $D_0(t)$. For intermediary times a transient decrease of $D(t)$ appears. This is determined by the magnetic line trapping around stochastic magnetic islands, which is effective at times larger than the flight time over the perpendicular correlation length λ_{\perp} , which in the unit considered here is $\tau_{\text{fl}} = 1/M$. The decay of $D(t)$ stops at $t = \tau_{\text{r}}$, the average return time for the parallel motion. At this moment $D(t)$ has a minimum while $D_0(t)$ has a maximum. Both the minimum and the maximum are determined by the parallel motion and more exactly by the collisions which force the particles to return on the magnetic lines. In the absence of the magnetic line trapping (quasilinear conditions) this leads to the decay of the running diffusion coefficient because the perpendicular displacement decreases in time and thus $D_0(t)$ decays at $t > \tau_{\text{r}}$. The effect is inverse in the presence of stochastic magnetic islands. The backward motion produces first the unmixing of the contribution of the trajectories that evolve on trapped magnetic lines. As time increases, the contributions of smaller and smaller magnetic islands are recovered in the Lagrangian velocity correlation. The effect of magnetic line trapping that produced the decay of $D(t)$ in the interval $(\tau_{\text{fl}}, \tau_{\text{r}})$ is washed out by the backward motion until $D(t)$ recovers its value at $t \sim \tau_{\text{fl}}$. At this moment τ_{b} , the correlation built-up time, $D(t)$ has a maximum. A positive bump appears in the Lagrangian velocity correlation due to the trajectories unwinding around the magnetic islands. The effect of stochastic magnetic islands disappears and the asymptotic regime in the evolution of the diffusion coefficient $D(t)$ is the same as for $D_0(t)$. Thus, the parallel collisional motion eliminates

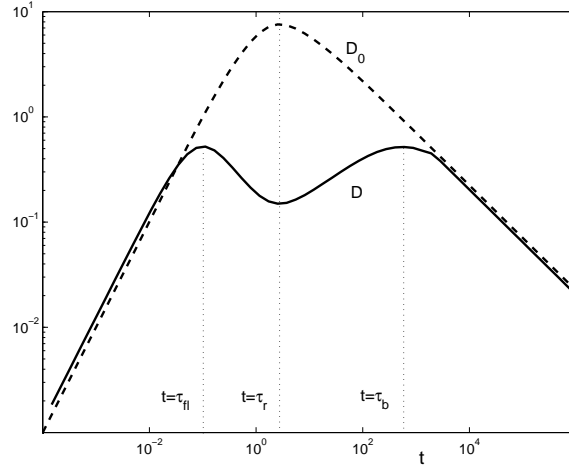


Figure 2. The time dependent diffusion coefficient $D(t)$ calculated for $M = 10$, $\bar{\chi}_\perp = 0$ and the quasilinear solution $D_0(t)$ valid for $M \ll 1$

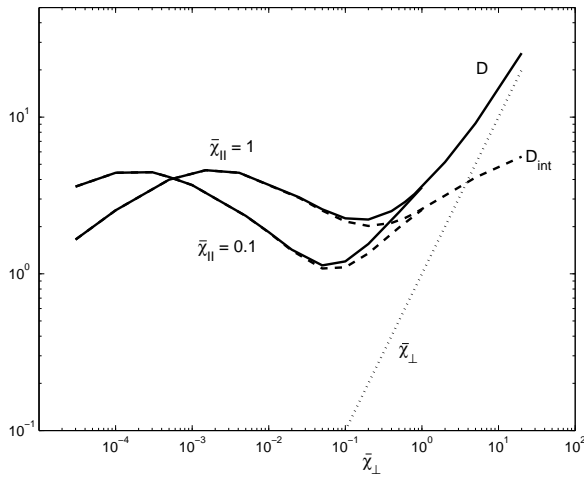


Figure 3. The asymptotic diffusion coefficient and its components D_{int} and $\bar{\chi}_\perp$ for $M = 10$

asymptotically the nonlinearity determined by the \mathbf{x} -dependence of the magnetic field fluctuations.

This rather nontrivial evolution of the running diffusion coefficient leads to anomalous diffusion regimes when the decorrelation of the particles from the magnetic lines induced by the cross field diffusivity $\bar{\chi}_\perp$ is considered. The perpendicular diffusion $\bar{\chi}_{\text{bot}}$ produces a releasing effect both for perpendicular and parallel components of particle motion. The transport is diffusive and the effective diffusion coefficient is anomalous in the presence of stochastic magnetic islands provided that $\bar{\chi}_\perp$ is small. Some results are presented in Fig. 4 where the asymptotic diffusion coefficient D is represented as a function of $\bar{\chi}_\perp$. The effective diffusion coefficient is obtained as the sum of the collisional diffusion and a term that contains the effects of the stochastic magnetic field influenced by collisions through the nonlinearity: $D = \bar{\chi}_\perp + D_{\text{int}}$. The two components D_{int} and $\bar{\chi}_\perp$ are also represented in Fig. 4. One can see that at small collisional diffusion $\bar{\chi}_\perp \ll 1$, the non-linear interaction term largely dominates the collisional term while at large collisional diffusion $\bar{\chi}_\perp \gtrsim 1$,

the nonlinear term is only a correction to $\bar{\chi}_\perp$. Thus, the subdiffusive transport appearing at $\bar{\chi}_\perp = 0$ is transformed by a small collisional cross field diffusion into a diffusive transport with a diffusion coefficient that can be several orders of magnitude larger than $\bar{\chi}_\perp$. The dependence of the diffusion coefficient on $\bar{\chi}_\perp$ is rather nontrivial. There is at very small $\bar{\chi}_\perp$ an increase of D up to a maximum which corresponds to $\tau_\perp \cong \tau_b$. Then, at larger $\bar{\chi}_\perp$, the nonlinear interaction of the parallel and perpendicular trapping with the collisional decorrelation generates a strange transport regime, in which the effective diffusion coefficient decreases as the collisional diffusion $\bar{\chi}_\perp$ increases. A minimum of D is obtained when $\bar{\chi}_\perp$ determines a decorrelation time $\tau_\perp = (2\bar{\chi}_\perp)^{-1}$ of the order of the return time of the parallel motion, $\tau_\perp \cong \tau_r$. At larger $\bar{\chi}_\perp$ (when $\tau_\perp < \tau_r$), the nonlinear contribution D_{int} increases again with the increase of $\bar{\chi}_\perp$ but this contribution begins to be comparable and eventually negligible compared to the collisional diffusion coefficient.

Thus, this rather complicated triple stochastic process (5.1)–(5.2) is characterized by two kinds of trajectory trappings and contains a decorrelation mechanisms. The latter are produced by the collisional cross field diffusion $\bar{\chi}_\perp$. One of the trapping processes concerns the parallel motion and is determined by collisions which constrain the particles to return in the already visited places with probability one. This parallel trapping leads to a subdiffusive transport in the absence of a decorrelation mechanism. The second kind of trapping concerns the magnetic lines which at $K_m > 1$ winds around the extrema of the potential generating self-consistently magnetic islands. The effective diffusion coefficient and its dependence on the parameters results from a competition between the trapping and the decorrelation processes and more precisely from the temporal ordering of the characteristic times of these processes. The presence of stochastic magnetic islands (at $K_m > 1$) produces a complicated nonlinear interaction between the three stochastic processes which determines new scaling laws of the diffusion coefficient. They appear when the decorrelation time is longer than the flight time τ_{fl} but smaller than the correlation built up time τ_b . The first condition ensures the magnetic islands generation and the second prevents the elimination of their trapping effect by the parallel collisional motion. A particularly interesting regime consists of an effective diffusion coefficient that decreases when the collisional perpendicular diffusion increases.

6. Transport of high energy particles

The generally accepted idea is that at large Larmor radius the test particle transport in stochastic magnetic fields is reduced due to the fact that the effective motion of the guiding centers is determined by the average of the stochastic potential on the cyclotron gyration. In recent papers (Vlad & Spineanu 2005, Vlad et al. 2005) it was shown that, in some specified conditions, the reversed effect can appear: the diffusion coefficient of high energy particles with Larmor radius $\rho > \lambda_c$, can be much larger than in the guiding center approximation.

We have consider the transport of particles with arbitrary Larmor radius in the stochastic magnetic field (4.1) with the average magnetic field large enough such that $\Omega\tau_{\text{fl}} \gg 1$, where $\Omega = qB_0/m$ is the cyclotron frequency in the average magnetic field, q is the charge of the particle and m its mass. The decorrelation trajectory method was extended and adapted to the transport of particles acted by the Lorentz force. We consider collisionless particles that have constant velocity along \mathbf{B}_0 . The results show that the diffusion coefficient strongly depends on $\bar{\rho} \equiv \rho/\lambda_c$ and K_m . At fixed value of the magnetic Kubo number, $D(\bar{\rho})$ has a maximum and then it decays as $\bar{\rho}^{-1}$. As seen in Fig. 5, for $K_m \leq 1$ the maximum of D is at $\bar{\rho} = 0$ and as K_m increases the maximum is shifted toward large values of $\bar{\rho}$. Also, at large K_m , the diffusion coefficient for small Larmor radius D_0

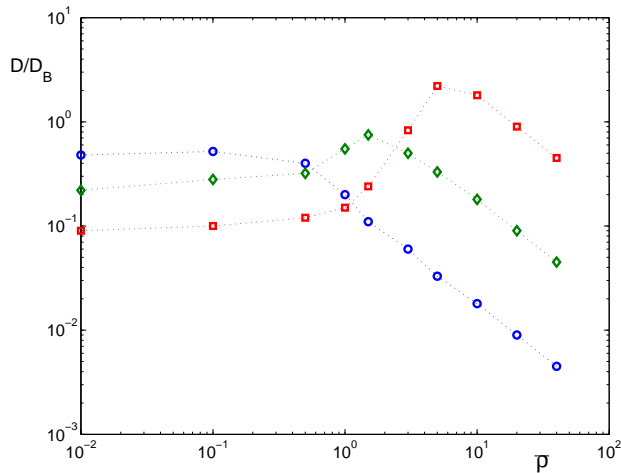


Figure 4. The diffusion coefficient D/D_B as a function of the Larmor radius $\bar{\rho}$ for $K = 1$ (circles), $K = 10$ (diamonds) and $K = 100$ (squares). $D_B = v_{\parallel}\beta\lambda_c$.

is reduced due to the trapping process that becomes effective at $K_m > 1$. For a range of values of $\bar{\rho}$ that increases with the increase of K_m , the diffusion coefficient is larger than D_o and its maximum value is much larger than D_o . The maximum of the diffusion coefficient at fixed Larmor radius appears at the “resonance” condition $K_m = 2\pi\bar{\rho}^2$.

We have shown that the quasilinear regime with the diffusion coefficient linear in the Kubo number extends up to large values of K_m , $K_m = 2\pi\bar{\rho}^2$. Due to this, transport coefficients much larger than D_o (the drift transport coefficient) can be obtained. The condition is that the stochastic magnetic field has a large but finite magnetic Kubo number. The increased diffusion of energetic particles is a nonlinear effect determined by the fast gyration motion that averages the stochastic potential leading to a strongly modified shape of the space dependence of the effective correlation. Magnetic line trapping and stochastic magnetic islands produce the trapping of the guiding center trajectories but with characteristics that are different from those of the drift transport.

7. Conclusions

We have discussed the problem of particle transport in stochastic magnetic fields. This is a special type of stochastic advection for which the nonlinear effects are very strong. The statistics of the magnetic lines is non-Gaussian, there is statistical memory and coherence at large magnetic Kubo numbers. These statistical properties of the magnetic lines are shown to be associated with structure formation in stochastic magnetic field that consists in stochastic magnetic islands oriented along the average magnetic field. The stochastic magnetic islands strongly influence the transport of collisional particles determining anomalous diffusion regimes with the diffusion coefficient much larger than the collisional diffusivity and than the collisionless stochastic diffusion coefficient. The diffusion produced by the Lorentz force for energetic particles with large Larmor radius was also studied. It was shown that the diffusion coefficient can be much larger than at small Larmor radius if the stochastic magnetic field has large magnetic Kubo number.

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