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Field theoretical methods in the theory of turbulence relaxation

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Abstract. We develop a field theoretical formalism for the dynamics of the point-like vortex models of the ideal fluid (Euler) equation and the Charney–Hasegawa–Mima equation for planetary atmosphere and magnetized plasma. The action functional in this framework exhibits the particular Bogomolnyi type extremum, showing that the fluids and plasmas relaxes to states that are self-dual. A new equation is derived. The solutions compares very well with observations, experiment and numerical simulations asymptotic stationary states in fluids atmosphere and plasma.

Keywords. Plasma, turbulence, magnetohydrodynamics

1. Introduction

Many experimental observations and numerical studies have shown that there is an intrinsic trend to self-organization of fluids and plasmas. The most evident manifestation appears at relaxation from turbulent initial states. For the ideal fluid described by the Euler equation (the nondissipative incompressible Navier–Stokes equation) it is known that the asymptotic state consists of few vortices animated by a slow residual motion. Actually at the limit there are only two vortices (positive and negative) formed after a continuous process of like-vortex merging.

Numerical simulations have initially suggested that the asymptotic states consists of few, distant, moving independently, vortices generated after a long process of like-vortex mergings. Actually if the numerical simulations is extended over very long time the true asymptotic state is attained and this consists of only two (one positive and one negative) vortices. Usually the emergence of ordered states from turbulence is considered to be a manifestation of the two-dimensional inverse energy cascade (for a review, see Kraichnan & Montgomery 1980). This consists of transfer of the energy by the nonlinear coupling of modes from small spatial scales to large spatial scales where it is supposed that there is an infrared energy sink, controlling the absorbtion of the spectral energy flux. The same process is sometimes described in terms of a negative eddy viscosity. Simultaneously there is a transfer of enstrophy (the square of the vorticity) from large spatial scales to small spatial scales and a fragmentation of the initial enstrophy into small fluctuations. This cascade is direct, according to standard classification. The process of transfer of energy and enstrophy, along the spectrum is the particular expression of the fluid self-organization and generation of coherent structure. It is interesting to note that representing the fluid as a set of point-like vortices (see below) the tendency toward self-organization is manifested as the negative temperature of the system.

The important role in this evolution of the dissipative mechanism is also a matter of careful consideration. For neutral fluids this is the molecular diffusivity related to collisions and for plasma this is the electric resistivity and the kinetic viscosity. The dissipation takes place in the region of high k (wavelength) space, at the end of the decay of the spectrum according to a typical Kolmogorov-type form, $k^{-\alpha}$ where α is the decay exponent of the spectrum. However in the process of generation of coherent structures in the asymptotic regime of fluid relaxing from turbulence the dissipative mechanism plays a very particular role. An ideal fluid (or plasma) has an infinite set of invariants as for example the integral of an arbitrary power of vorticity over a surface limited by a streamline, $\psi = \text{const}$. The streamlines in an ideal fluid are completely defined by the initial configuration, they simply evolve by homotopic deformations, which means they cannot be broken and reconnected. However the role of a dissipative mechanism is precisely to allow breaking and reconnection of streamlines (or magnetic field lines in plasma). In this way the system (fluid or plasma) can access states that are more "convenient" (for example of lower energy). Certainly in any reconnection event there is some energy loss via dissipation. Since these events occur as localized and very fast processes the total energy loss is insignificant over the range of time for reaching asymptotic states. The resistivity and the viscosity simply provide the possibility of reconfigurations allowing the system to evolve to more convenient states (see below) without affecting in a significant way the energy content. This is supported by numerical simulations. In plasma the tearing instability evolves due to a finite resistivity strictly limited in the region of null magnetic field and is active on only the duration of reconnection (Biskamp 2001). Except this space-time-localized event, the plasma can be considered ideal. It is known that in a volume of fluid the dissipative mechanism is active for very short intervals of time and in regions which represent together a fractal subset in space. This actually corresponds to the regions where reconnection of flow lines occurs. It is also known from numerical simulation that the system exploits the possibility to reorganize by reconnections of streamlines at early times in relaxation while after reaching the state of equilibrium it simply remains in this quasi-stationary state for very long time, until the dissipation transforms all energy in thermal energy reducing all motion to zero (see Horton, Tajima & Kamimura 1987, Kinney, McWilliams & Tajima 1995, Kinney et al. 1994, Horton & Hasegawa 1994).

There is a considerable effort to understand the nature of the structure formation in the evolution of the system. Two hypothesis have been formulated: selective decay of ideal invariants and dynamic alignment of the orbits on which particular quantities are invariant. This approach is closely related with the classical conception about turbulence as a spectral dynamical process.

In this work we will develop a completely different perspective. For this we will first place different degrees of significance on the same objects used in the classical approach: we will be guided by the idea that the stationary state attained by the fluid/plasma represents a state of highest symmetry in a theory where *matter* and *field* are present, and this corresponds to the equality of the two-form gauge field stress with its two-form dual, or, the *self-dual states*.

2. The ideal fluid: the Euler equation

Inferring from results of numerical simulations of the relaxation in Euler fluids Montgomery and co-workers (Montgomery et al. 1992, Fyfe, Montgomery & Joyce 1976, Joyce & Montgomery 1973, Montgomery, Turner & Vahala 1979) have proved that the scalar stream function ψ describing the motion in two-dimensional space obeys in the far asymptotic regime (where the regular structures are dominant) the sinh-Poisson equation

$$\Delta\psi + \gamma \sinh(\beta\psi) = 0 \tag{2.1}$$

where γ and β are *positive* constants. The relations of ψ to the velocity and vorticity are $\mathbf{v} = \nabla\psi \times \widehat{\mathbf{e}}_z$, $\omega = \nabla \times \mathbf{v} = -\nabla^2\psi\widehat{\mathbf{e}}_z$ where $\widehat{\mathbf{e}}_z$ is the unitary vector perpendicular to the plane. With these variables, the Euler equations for the two dimensional ideal incompressible fluid are

$$\nabla \cdot \mathbf{v} = 0, \quad \frac{\partial \omega}{\partial t} + (\mathbf{v} \cdot \nabla)\,\omega = \mathbf{0} \tag{2.2}$$

In the study of the two-dimensional Euler fluids, and in particular in explaining the origin of Eq. (2.1), an important model consists of a system of N discrete vorticity filaments perpendicular on plane, having circular transversal section of radius a and carrying the vorticity ω_i, $i = 1, N$ (Kraichnan & Montgomery 1980). The motion in plane of the k-th filament of coordinates $\mathbf{r}_k \equiv \left(r_k^1, r_k^2\right) \equiv (x_k, y_k)$ is given by

$$\frac{dr_k^i}{dt} = \varepsilon^{ij}\frac{\partial}{\partial r_k^j}\sum_{n=1,n\neq k}^{N}\omega_n G\left(\mathbf{r}_k - \mathbf{r}_n\right)\ ,\ i,j = 1,2\ ,\ k = 1, N \tag{2.3}$$

where the summation is over all the other filaments' positions $\mathbf{r}_n$, $n \neq k$, and ε^{ij} is the antisymmetric tensor in two dimensions. $G\left(\mathbf{r}_k - \mathbf{r}_n\right)$ can be written for $a \ll L$ as the Green function of the Laplacian $G\left(\mathbf{r}, \mathbf{r}'\right) \approx -\frac{1}{2\pi}\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L}\right)$. If we take equal strength ω for all vortices this potential appears as the "statistical potential" and has a topological interpretation $\frac{1}{2\pi}\varepsilon^{ij}\frac{r^j}{r^2} = -\frac{1}{2\pi}\partial_i \arctan\frac{y}{x} = -\frac{1}{2\pi}\partial_i\theta$ where $\mathbf{r} = (x, y) = (r\cos\theta, r\sin\theta)$. The "magnetic" flux through a surface limited by a large circle is proportional with the number of vortices, which can be seen as a topological charge. This suggests that it can be naturally derived from a Lagrangian density of the Chern–Simons type,

$$\mathcal{L} = \frac{1}{4}\varepsilon^{\mu\alpha\beta}A_\nu F_{\alpha\beta} \tag{2.4}$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. The vorticity ω is the density ρ of point-like vortices and is expressed in terms of a complex field $\Psi \in SU(2)$ (Dunne 1994, Jackiw & Pi 1990a), $\rho \sim \left[\Psi^\dagger, \Psi\right]$. The Lagrangian density for our model becomes

$$\begin{aligned}\mathcal{L} = -\varepsilon^{\mu\nu\rho}Tr\left(\partial_\mu A_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) + \\ iTr\left(\Psi^\dagger D_0\Psi\right) - \frac{1}{2}Tr\left(\left(D_i\Psi\right)^\dagger D_i\Psi\right) + \frac{1}{4}Tr\left(\left[\Psi^\dagger, \Psi\right]\right)^2\end{aligned} \tag{2.5}$$

where the potential A takes values in the algebra of the group $SU(2)$ and $D_\mu\Psi = \partial_\mu\Psi + [A_\mu, \Psi]$. The equations of motion are

$$iD_0\Psi = -\frac{1}{2}\mathbf{D}^2\Psi - \frac{1}{2}\left[\left[\Psi, \Psi^\dagger\right], \Psi\right] \tag{2.6}$$

$$F_{\mu\nu} = -\frac{i}{2}\varepsilon_{\mu\nu\rho}J^\rho \tag{2.7}$$

Using the notation $D_\pm \equiv D_1 \pm iD_2$ the energy density is $\mathcal{H} = \frac{1}{2}Tr\left(\left(D_-\Psi\right)^\dagger\left(D_-\Psi\right)\right) \geqslant 0$ and the Bogomol'nyi inequality is saturated at *self-duality*

$$D_-\Psi = 0 \tag{2.8}$$

$$\partial_+A_- - \partial_-A_+ + [A_+, A_-] = \left[\Psi, \Psi^\dagger\right] \tag{2.9}$$

The *static* solutions of the *self-duality* equations (2.8, 2.9) are derived in Dunne 1994, using the algebraic *Ansatz*:

$$A_i = \sum_{a=1}^{r} A_i^a H_a, \Psi = \sum_{a=1}^{r} \psi^a E_a + \psi^M E_{-M} \tag{2.10}$$

where H_a are the Cartan subalgebra generators for the gauge Lie algebra, E_a are the simple-root step operators and E_{-M} is the step operator corresponding to minus the maximal root. The rank of the algebra is noted r, and $r = 1$ for $SU\,(2)$. Then $\left[\Psi^\dagger, \Psi\right] = \sum_{a=1}^{r} |\psi^a|^2 H_a + \left|\psi^M\right|^2 H_{-M}$. The equations (2.8, 2.9) lead to the affine Toda equations

$$\nabla^2 \ln \rho_a + \sum_{b=1}^{r+1} \widetilde{C}_{ab}\rho_b = 0 \tag{2.11}$$

for $a = 1, r$, plus the index for M, i.e., $a = 1, 2$. $\widetilde{C}_{ab}$ is the extended Cartan matrix $\widetilde{C}_{ab} = 2\alpha^{(a)} \cdot \alpha^{(b)} / \left|\alpha^{(b)}\right|^2$, $a, b = 1, 2$ where $\alpha^{(a)}$ are the simple root vectors of the algebra $su\,(2)$, and in addition the minus maximal root. The equations (2.11) can be written in detail for $\rho_1 \equiv \left|\psi^1\right|^2$, $\rho_2 \equiv \left|\psi^{-M}\right|^2$

$$\begin{aligned} \Delta \ln \rho_1 + 2(\rho_1 - \rho_2) = 0 \\ \Delta \ln \rho_2 + 2(-\rho_1 + \rho_2) = 0 \end{aligned} \tag{2.12}$$

and this gives the relation $\Delta \ln(\rho_1\rho_2) = 0$ or $\rho_2 =$const ρ_1^{-1} for the simplest choice of solution to the Laplace equation. We have to identify

$$\omega = \rho_1 - \rho_2 \tag{2.13}$$

Taking const $= 1$ we have

$$\Delta \ln \rho_1 + 2(\rho_1 - \rho_1^{-1}) = 0 \tag{2.14}$$

The substitution $\psi' \equiv \ln \rho_1$ transforms Eq. (2.14) into $-\Delta\,(\psi'/2) = 2\sinh(\psi')$ and Eq. (2.13) into $\omega = 2\sinh(\psi')$. Actually we can multiply ψ with an arbitrary constant γ and/or put in front of sinh any other arbitrary constant β since these can be absorbed in scalings of the space variables (x, y). We then have

$$\Delta\psi + \gamma \sinh(\beta\psi) = 0. \tag{2.15}$$

This is a purely analytic derivation of the sinh-Poisson equation as the equation governing the asymptotic stationary states of (quasi) ideal Euler fluids (Spineanu & Vlad 2003). The derivation has been made possible by the existence of the mapping between the physical Euler equation and the model of point-like vortices in plane. The latter model seems useful just as a discrete form of the Euler fluid, however we have considered the continuum version, letting the number of point vortices to go to infinity. This should return us to the starting point, however something fundamentally new appears: the theory is now formulated in terms of matter, gauge field and interaction. It is this fact that allows us to formalize as a field theory, with a Lagrangian density containing matter, gauge field and interaction.

3. The planetary atmosphere and the magnetized plasma: the Charney–Hasegawa–Mima equation

The drift waves in two-dimensional confined plasma and the Rossby waves in atmosphere are described by Charney–Hasegawa–Mima equation (Charney 1948, Hasegawa

& Mima 1979) for the function ϕ (the streamfunction for the Rossby wave and the electrostatic potential for the plasma drift wave, see Horton & Hasegawa 1994)

$$\left(1-\nabla_{\perp}^{2}\right)\frac{\partial\phi}{\partial t}-v_{*}\frac{\partial\phi}{\partial y}-\left[\left(-\nabla_{\perp}\phi\times\widehat{\mathbf{n}}\right)\cdot\nabla_{\perp}\right]\nabla_{\perp}^{2}\phi=0. \tag{3.1}$$

Restricting to plasma terminology ϕ is the electrostatic potential, $\widehat{\mathbf{n}}$ is the direction of the magnetic field, (x, y) are the radial and poloidal coordinates in a plasma confinement geometry (like in tokamak) and $v_{*}\widehat{\mathbf{e}}_y = -\widehat{\mathbf{n}}\times\nabla_{\perp}\ln n_0$, with $n_0(x)$ the density and $\widehat{\mathbf{e}}_y$ the versor along y. The physical (superscript "phys") variables are normalized as: $\phi = |e|\,\phi^{\text{phys}}/T_e$, $(x,y) = \left(x^{\text{phys}}/\rho_s,\ y^{\text{phys}}/\rho_s\right)$, $t = t^{\text{phys}}\Omega_{ci}$, where $\Omega_{ci} = |e|\,B/m_i$, $\rho_s = c_s/\Omega_{ci}$, $c_s^2 = T_e/m_i$. Here B is the magnetic field, $|e|$ and m_i are the ion electric charge and mass, T_e is the electron temperature.

With very low viscosity the plasma evolves to states of organized flow consisting of few vortices with regular shape. There is a discrete vortex model for the CHM equation as well. It has been proposed b Stewart (1943) and Morikawa (1960) and consists of a discrete set of N point-like vortices with vorticity ω_k with the equations of motion

$$\begin{aligned}-2\pi\omega_k\frac{dx_k}{dt}&=\frac{\partial W}{\partial y_k}\\ -2\pi\omega_k\frac{dy_k}{dt}&=-\frac{\partial W}{\partial x_k}\end{aligned} \tag{3.2}$$

In the equations of motion the potential is $W = \pi\sum_{j\neq i}^{N}\omega_i\omega_j K_0\left(m\left|\mathbf{r}_i-\mathbf{r}_j\right|\right)$ and K_0 is the modified Bessel function. There is an intrinsi spatial scale of the CHM equation $\rho_s = m^{-1}$. A field theoretical version of the continuum limit of the point-like vortices should contain (1) a term for the free gauge field that produces the potential of interaction between the vortices; (2) terms for the matter (related to the density of vortices); there will be terms for the kinematic part and for the nonlinear self-interaction; (3) minimal coupling between gauge and scalar fields, via the covariant derivatives. Since the scalar field ϕ results from the density of positive and negative vortices we note that the elementary vortices have much in common with complex Weyl spinors. It is then appropriate to work in the most general formulation, in which the fields ϕ and A_μ belong to the adjoint representation of the $SU(2)$ algebra. Then the Lagrangian density has the expression (see Jackiw, Lee & Weinberg 1990, Dunne et al. 1991, Hong, Kim & Pak 1990, Jackiw & Weinberg 1990, Jackiw & Pi 1990b)

$$\begin{aligned}\mathcal{L}=&-\kappa\varepsilon^{\mu\nu\rho}\operatorname{tr}\left(\partial_\mu A_\nu A_\rho+\frac{2}{3}A_\mu A_\nu A_\rho\right)\\&-\operatorname{tr}\left[\left(D^\mu\phi\right)^{\dagger}\left(D_\mu\phi\right)\right]\\&-V\left(\phi,\phi^{\dagger}\right)\end{aligned} \tag{3.3}$$

with

$$\begin{aligned}&V\left(\phi,\phi^{\dagger}\right)\\&=\frac{1}{4\kappa^2}\operatorname{tr}\left[\left(\left[\left[\phi,\phi^{\dagger}\right],\phi\right]-v^2\phi\right)^{\dagger}\left(\left[\left[\phi,\phi^{\dagger}\right],\phi\right]-v^2\phi\right)\right].\end{aligned} \tag{3.4}$$

This is the $(2+1)$-dimensional field theoretical framework for the continuum limit of the CHM point-like vortex model. Here $D_\mu = \partial_\mu + [A_\mu,]$ and the symbol † mean Hermitian conjugate. The Euler–Lagrange equations are

$$D_\mu D^\mu \phi = \frac{\partial V}{\partial \phi^\dagger}$$
$$-\kappa \varepsilon^{\nu\mu\rho} F_{\mu\rho} = iJ^\nu$$

where the field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The current is

$$J^\mu = -i\left(\left[\phi^\dagger, D^\mu\phi\right] - \left[(D^\mu\phi)^\dagger, \phi\right]\right) \tag{3.5}$$

and the Gauss law constraint is $-2\kappa F_{12} = iJ^0$ or $2\kappa F_{12} = \left[\phi^\dagger, D_0\phi\right] - \left[(D_0\phi)^\dagger, \phi\right]$. Very detailed calculations are presented in Spineanu & Vlad 2005a. The action functional for the Lagrangian density (3.3) can be written in the Bogomolnyi form (Dunne et al. 1991), from which one derives that the extremum of the action is realized by the states verifying the self-duality equations

$$D_-\phi = 0 \tag{3.6}$$
$$F_{+-} = \frac{1}{\kappa^2}\left[v^2\phi - \left[\left[\phi, \phi^\dagger\right], \phi\right], \phi^\dagger\right]$$

where $D_- \equiv D_1 - iD_2$, $A_\pm \equiv A_1 \pm iA_2$, $F_{+-} \equiv \partial_+ A_- - \partial_- A_+ - [A_+, A_-]$.

As suggested in previous works (Dunne et al. 1991, Dunne 1999), the following algebraic*Ansatz* can be adopted

$$\phi = \phi_1 E_+ + \phi_2 E_- \ , \ \phi^\dagger = \phi_1^* E_- + \phi_2^* E_+,$$
$$A_+ = aH \ , \ A_- = -a^* H.$$

For this rank 1 algebra the Chevalley basis is $\{E_\pm, H\}$ with $[E_+, E_-] = H$, $[H, E_\pm] = \pm 2E_\pm$, $\mathrm{tr}\,(E_+E_-) = 1$, $\mathrm{tr}\,(H^2) = 2$. Using the *Ansatz* and introducing the notations $\rho_1 \equiv |\phi_1|^2$, $\rho_2 \equiv |\phi_2|^2$ we obtain as in the Euler fluid case $\Delta \ln(\rho_1\rho_2) = 0$. Taking simply $\rho_1\rho_2 = v^4/\left(16p^2\right)$, with p a positive constant we have

$$\Delta \ln \rho = -\frac{1}{4p^2}\left(\frac{v^2}{\kappa}\right)^2 \left(\rho - \frac{1}{\rho}\right)\left[\frac{1}{2}\left(\rho + \frac{1}{\rho}\right) - p\right] \tag{3.7}$$

$$\Delta\psi + \frac{1}{2p^2}\sinh\psi\,(\cosh\psi - p) = 0 \tag{3.8}$$

This is the equation governing the stationary asymptotic states of the CHM equation. We conclude that the mass of the photon is

$$m = \frac{v^2}{\kappa} \tag{3.9}$$

and this mass is generated via the Higgs mechanism adapted to the Chern–Simons action (see review by Dunne 1999). The photon acquires a mass because it moves in a background where the scalar field is equal with the vacuum value, v^2, which is not zero. This mass induces the short spatial range of the interaction in the discrete vortices model, introduced by Stewart and Morikawa in meteorology. We have $m = v^2/\kappa = 1/\rho_s$. In physical terms $\kappa \equiv c_s$, $v^2 \equiv \Omega_{ci}$. The vortical flows of the CHM equation are excitations over the background of "vorticity" represented by the Larmor gyration, intrinsically present in the CHM equation.

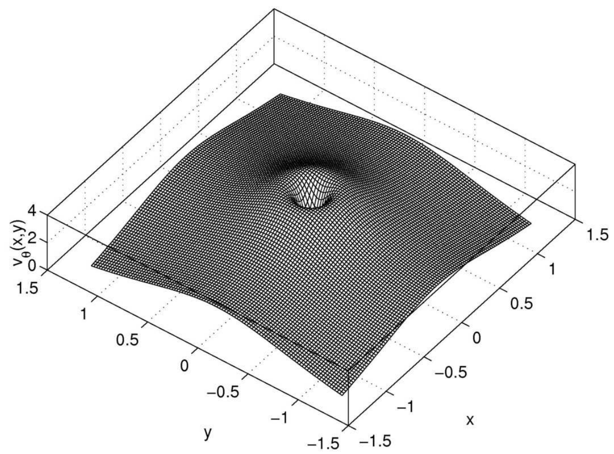

Figure 1. Azimuthal velocity for a cyclone with radius of 1.25× (Rossby radius).

4. Applications

There are very favorable comparisons with scatterplots obtained in experiments (see deRooij, Linden & Dalziel 1999) or numerical simulations (Seyler 1996, Li, Montgomery & Jones 1997). The analytical solution is probably not available via the Inverse Scattering Transform (Dodd & Bullough 1976). We have implemented the code GIANT (Nowak & Weimann 1990) and have carried out a very large number of numerical experiments. They can be classified according to the domain of variation of the parameters and this corresponds rather well to several types of vortices observed in nature or laboratory.

In the field-theory action functional there are only two physical parameters: the coefficient of the Chern–Simons action κ, which we have identified as the sound speed, c_s and the asymptotic vorticity v^2 which is the Coriolis frequency f_0 (or, in plasma physics, the ion cyclotron frequency Ω_{ci}). The space-like parameter that normalizes the Laplace operator in the Eq. (3.8) is the ratio

$$\rho_g = c_s/f_0 \tag{4.1}$$

i.e., the Rossby radius ρ_g (respectively the sonic Larmor radius in plasma, $\rho_s = c_s/\Omega_{ci}$). All distances implied in the solution are normalized ρ_g. The streamfunction is normalized as (the upperscript "phys" is used to indicate that the quantity is dimensional)

$$\psi = \frac{\psi^{\text{phys}}}{\rho_g^2 f_0} \tag{4.2}$$

The unit for the stream function is $\rho_g^2 f_0$ and the unit for vorticity f_0. Then the unit for velocity is $\rho_g f_0$.

If we know the large radial extension of a vortex (L^{phys}) the normalized parameter L is obtained by dividing to ρ_g. The result of integration is very sensitive to L and this points out the essential role played by ρ_g in numerical studies aiming to reproduce observations. We initialize the iterations of GIANT by a vortex whose spatial profile is $[\text{sech}\,(kr)]^{4/3}$ (k is a constant), similar to the Pokhotelov–Petviashvili solution to the Flierl–Petviashvili equation.

The solutions to the differential equation have the same morphology as the two-dimensional flow of a tropical cyclone. In Spineanu & Vlad 2005b solutions are compared with Willoughby & Black 1996, Wang & Wu 2004, Reasor & Montgomery 2001 and Kossin & Schubert 2001. The solutions are characterized by a very narrow dip in the azimuthal velocity (tangential wind) in the center of the vortex. The radius of the "maximum tangential wind" or the radius of the *eye wall* is much smaller than the radius of the vortex. There is a decay of the magnitude of the azimuthal velocity toward the periphery, which is much slower compared with the fast decay toward the center. We find a very low magnitude (almost vanishing) of the vorticity over most of the vortex (approx. from the radius of maximum wind to the periphery), while the magnitude in a narrow central region is extremely high. Quantitatively, we obtain for the diameter of the cyclone's eye a magnitude which compares well with the observations. The maximum vorticity is in a realistic range and the radial profile of the tangential velocity is similar to what is found in observations or with what is obtained in empirical models and numerical simulations.

We have inferred from numerical data an expression showing variation of $v_\theta^{\max}$ (the maximum tangential wind)

$$v_\theta^{\max}(L) \simeq \frac{e^2}{2}\left[\alpha \exp\left(\frac{1}{L}\right) - 1\right] \tag{4.3}$$

for the interval $0 < L < 6$. The constant is $\alpha \simeq 0.97$.

Another application of this equation consists of comparison of its solutions with the experimentally observed vortex in the linear plasma device HYPER-I (NIFS, Japan) Nagaoka et al. 2002. The range of parameters results from normalization with: time Ω_{ci}^{-1}, velocity c_s, and distances $\rho_i = c_s/\Omega_{ci}$. The numerical solutions reproduce well the profiles of measured vorticity.

Finally we have carried out numerical studies for vortex crystals as they have been produced in the non-neutral plasma. The equation is able to produce quasi-solutions with the symmetry and dimensions similar to the one observed in experiments. These are not exact solution (although the code finds they are very close of being solutions) and they eventually evolve to symmetric vortices, like in real electron plasma.

In conclusion we have presented a field theoretical framework for the point-like vortices models of two dimensional plasma and atmosphere. We have shown that the extremum of the action corresponds to stationary self-dual states and we have found the differential equation governing these states. The comparison with the experiment and numerical simulation is favorable.

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