STABILITY OF CIRCULARLY POLARIZED ALFVÉN WAVES

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CONTENT

1. CIRCULARLY POLARIZED ALFVÉN WAVES

We use the system of ideal MHD equations: Mass conservation equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{1.1}$$

Momentum equation:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} + \frac{1}{\mu_0 \rho}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \tag{1.2}$$

Induction equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \tag{1.3}$$

Adiabatic equation:

$$p = p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma} \tag{1.4}$$

We introduce Cartesian coordinates x, y, z, and consider solutions of system (1.1)–(1.4) that are independent of y and z $\Longrightarrow$ we rewrite (1.1)–(1.4) as

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{1.5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial}{\partial x}\left(p + \frac{|\boldsymbol{B}_\perp|^2}{2\mu_0}\right) \tag{1.6}$$

$$\frac{\partial \boldsymbol{v}_\perp}{\partial t} + u\frac{\partial \boldsymbol{v}_\perp}{\partial x} = \frac{B_x}{\mu_0\rho}\frac{\partial \boldsymbol{B}_\perp}{\partial x} \tag{1.7}$$

$$\frac{\partial \boldsymbol{B}_\perp}{\partial t} = B_x\frac{\partial \boldsymbol{v}_\perp}{\partial x} - \frac{\partial(u\boldsymbol{B}_\perp)}{\partial x} \tag{1.8}$$

$$p = p_0\left(\frac{\rho}{\rho_0}\right)^\gamma \tag{1.9}$$

Here $\boldsymbol{v} = (u, v, w)$, $\boldsymbol{v}_\perp = (0, v, w)$, $\boldsymbol{B}_\perp = (0, B_y, B_z)$, $B_x = \text{const.}$

Let us consider such a solution of (1.5)–(1.9) that

$$\rho = \rho_0, \quad p = p_0, \quad u = 0, \quad |\boldsymbol{B}_\perp| = A = \text{const} \tag{1.10}$$

$\Longrightarrow$ Eqs. (1.5), (1.6) and (1.9) are satisfied automatically. Eqs. (1.7) and (1.8) are transformed to

$$\frac{\partial \boldsymbol{v}_\perp}{\partial t} = \frac{B_x}{\mu_0 \rho_0} \frac{\partial \boldsymbol{B}_\perp}{\partial x}, \quad \frac{\partial \boldsymbol{B}_\perp}{\partial t} = B_x \frac{\partial \boldsymbol{v}_\perp}{\partial x} \tag{1.11}$$

Eliminating $\boldsymbol{v}_\perp$ from (1.11) we obtain

$$\frac{\partial^2 \boldsymbol{B}_\perp}{\partial t^2} - V_A^2 \frac{\partial^2 \boldsymbol{B}_\perp}{\partial x^2} = 0, \quad V_A^2 = \frac{B_x^2}{\mu_0 \rho_0} \tag{1.12}$$

V_A is **Alfvén speed**. (1.10) $\Longrightarrow$ $B_y^2 + B_z^2 = A^2$ $\Longrightarrow$

$$B_y = A \cos\phi, \quad B_z = A \sin\phi, \quad \phi = \phi(t, x) \tag{1.13}$$

Substitute (1.13) in (1.12) $\Longrightarrow$

$$
\begin{aligned}
\sin\phi\,\frac{\partial^2\phi}{\partial t^2} + \cos\phi\left(\frac{\partial\phi}{\partial t}\right)^2 &= V_A^2\left[\sin\phi\,\frac{\partial^2\phi}{\partial x^2} + \cos\phi\left(\frac{\partial\phi}{\partial x}\right)^2\right] \\
\cos\phi\,\frac{\partial^2\phi}{\partial t^2} - \sin\phi\left(\frac{\partial\phi}{\partial t}\right)^2 &= V_A^2\left[\cos\phi\,\frac{\partial^2\phi}{\partial x^2} - \sin\phi\left(\frac{\partial\phi}{\partial x}\right)^2\right]
\end{aligned}
\qquad (1.14)
$$

(1.14) $\Longleftrightarrow$

$$
\frac{\partial^2\phi}{\partial t^2} = V_A^2\frac{\partial^2\phi}{\partial x^2}, \qquad \left(\frac{\partial\phi}{\partial t}\right)^2 = V_A^2\left(\frac{\partial\phi}{\partial x}\right)^2 \qquad (1.15)
$$

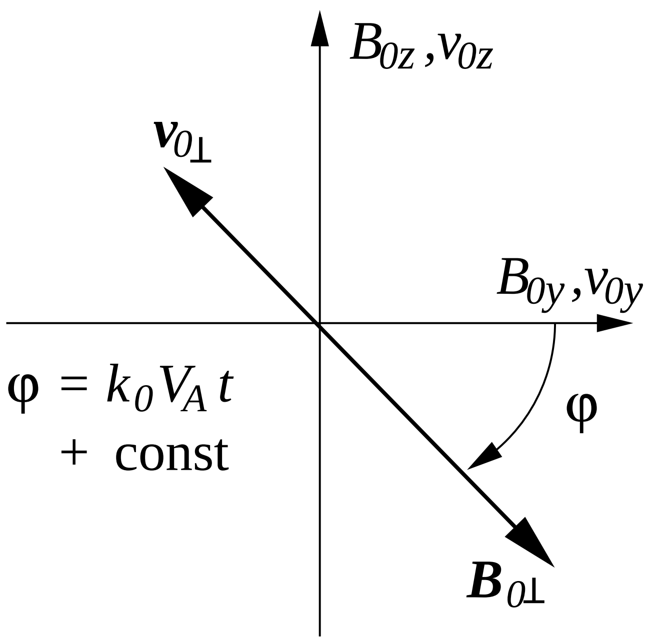
It follows from the second equation that

$$
\frac{\partial\phi}{\partial t} \pm V_A\frac{\partial\phi}{\partial x} = 0 \qquad (1.16)
$$

Sing $+$ $(-)$ corresponds to wave propagating in positive (negative) direction of x-axis. We consider wave propagating in positive direction $\Longrightarrow$ take $+$ in (1.16). Then (1.16) $\Longrightarrow$ $\phi = \phi(x - V_A t)$.

In what follows we take $\phi = k_0(x - V_A t)$. Then, using (1.11) and (1.13), we finally obtain $\boldsymbol{B}_\perp = \boldsymbol{B}_{0\perp}$, $\boldsymbol{v}_\perp = \boldsymbol{v}_{0\perp}$, where

$$
\begin{aligned}
&B_{0y} = A\cos[k_0(x - V_A t)], \qquad && B_{0z} = A\sin[k_0(x - V_A t)] \\
&v_{0y} = -\frac{AV_A}{B_x}\cos[k_0(x - V_A t)], && v_{0z} = -\frac{AV_A}{B_x}\sin[k_0(x - V_A t)]
\end{aligned}
\tag{1.17}
$$



We see that $\boldsymbol{B}_{0\perp}$ and $\boldsymbol{v}_{0\perp}$ are unti-parallel. At fixed x the both vectors rotate with angular velocity $k_0 V_A$ clockwise (anticlockwise) when $k_0 > 0$ ($k_0 < 0$).

2. DERIVATION OF DISPERSION EQUATION

We write all dependent variables in the form $f = f_0 + f'$, where f_0 represents circularly polarized Alfvén wave and f' is perturbation. Then we substitute these expressions in (1.5)–(1.9) and linearize the obtained equations, i.e. retain only terms linear with respect to f'. As a result we obtain

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = 0 \tag{2.1}$$

$$\frac{\partial u'}{\partial t} = -\frac{c_S^2}{\rho_0}\frac{\partial \rho'}{\partial x} - \frac{A}{\mu_0 \rho_0}\frac{\partial}{\partial x}(B'_y \cos\phi + B'_z \sin\phi) \tag{2.2}$$

$$\frac{\partial \boldsymbol{v}'_\perp}{\partial t} + u'\frac{\partial \boldsymbol{v}_{0\perp}}{\partial x} = \frac{B_x}{\mu_0 \rho_0}\frac{\partial \boldsymbol{B}'_\perp}{\partial x} - \frac{B_x \rho'}{\mu_0 \rho_0^2}\frac{\partial \boldsymbol{B}_{0\perp}}{\partial x} \tag{2.3}$$

$$\frac{\partial \boldsymbol{B}'_\perp}{\partial t} = B_x \frac{\partial \boldsymbol{v}'_\perp}{\partial x} - \frac{\partial (u' \boldsymbol{B}_{0\perp})}{\partial x} \tag{2.4}$$

Here $c_S^2 = \gamma p_0/\rho_0$ is the square of the sound speed.

System (2.1)–(2.4) has variable coefficients. Let us introduce new variables

$$
\begin{aligned}
B_+ &= B'_y \cos\phi + B'_z \sin\phi, \\
B_- &= B'_y \sin\phi - B'_z \cos\phi \\
v_+ &= v'_y \cos\phi + v'_z \sin\phi, \\
v_- &= v'_y \sin\phi - v'_z \cos\phi
\end{aligned}
\tag{2.5}
$$

In new variables system (2.1)–(2.4) takes the form

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = 0 \tag{2.6}$$

$$\frac{\partial u'}{\partial t} = -\frac{c_S^2}{\rho_0}\frac{\partial \rho'}{\partial x} - \frac{A}{\mu_0 \rho_0}\frac{\partial B_+}{\partial x} \tag{2.7}$$

$$\frac{\partial v_+}{\partial t} - k_0 V_A v_- = \frac{B_x}{\mu_0 \rho_0}\left(\frac{\partial B_+}{\partial x} + k_0 B_-\right) \tag{2.8}$$

$$\frac{\partial v_-}{\partial t} + k_0 V_A v_+ + \frac{A V_A u'}{B_x} = \frac{B_x}{\mu_0 \rho_0}\left(\frac{\partial B_-}{\partial x} - k_0 B_+ + \frac{A k_0 \rho'}{\rho_0}\right) \tag{2.9}$$

$$\frac{\partial B_+}{\partial t} - k_0 V_A B_- = B_x\left(\frac{\partial v_+}{\partial x} + k_0 v_-\right) - A\frac{\partial u'}{\partial x} \tag{2.10}$$

$$\frac{\partial B_-}{\partial t} + k_0 V_A B_+ = B_x\left(\frac{\partial v_-}{\partial x} - k_0 v_+\right) + A k_0 u' \tag{2.11}$$

This system has **constant coefficients!**

Now we take all dependent variables proportional to $\exp[i(Kx - \Omega t]$. As a result we obtain the system of algebraic equations:

$$\Omega\rho' - \rho_0 K u' = 0 \tag{2.12}$$

$$\Omega u' - K\left(\frac{c_S^2}{\rho_0}\rho' + \frac{A}{\mu_0\rho_0}B_+\right) = 0 \tag{2.13}$$

$$\Omega v_+ - ik_0 V_A v_- + \frac{B_x}{\mu_0\rho_0}(KB_+ - ik_0 B_-) = 0 \tag{2.14}$$

$$\Omega v_- + ik_0 V_A v_+ + \frac{iAV_A u'}{B_x} + \frac{B_x}{\mu_0\rho_0}\left(KB_- + ik_0 B_+ - \frac{iAk_0\rho'}{\rho_0}\right) = 0 \tag{2.15}$$

$$\Omega B_+ - ik_0 V_A B_- + B_x(Kv_+ - ik_0 v_-) - AKu' = 0 \tag{2.16}$$

$$\Omega B_- + ik_0 V_A B_+ + B_x(Kv_- + ik_0 v_+) + iAk_0 u' = 0 \tag{2.17}$$

(2.12)–(2.17) is a system of linear homogeneous algebraic equations. It has a non-trivial solution only when its determinant is zero. This condition given the **dispersion equation**:

$$\begin{aligned}&\Big\{(\Omega^2 - c_S^2 K^2)(\Omega - V_A K)\big[(\Omega + V_A K)^2 - 4V_A^2 k_0^2\big]\\ &\quad - (A/B_x)^2 V_A^2 K^2(\Omega^3 + V_A\Omega^2 K - 3V_A^2\Omega k_0^2\\ &\quad + V_A^3 k_0^2 K)\Big\}(\Omega - V_A K) = 0 \qquad (2.18)\end{aligned}$$

The second multiplier in the dispersion equation (2.18) gives the solution $\Omega = V_A K$ which does not lead to instability $\Longrightarrow$ we disregard this multiplier. Then, introducing the dimensionless variables

$$a = \frac{A}{B_x}, \quad b = \frac{c_S}{V_A}, \quad k = \frac{K}{k_0}, \quad \omega = \frac{\Omega}{V_A k_0},$$

we rewrite the dispersion equation as

$$D(k,\omega) \equiv (\omega^2 - b^2 k^2)(\omega - k)\left[(\omega + k)^2 - 4\right] - a^2 k^2(\omega^3 + \omega^2 k - 3\omega + k) = 0 \qquad (2.19)$$

3. STABILITY ANALYSIS

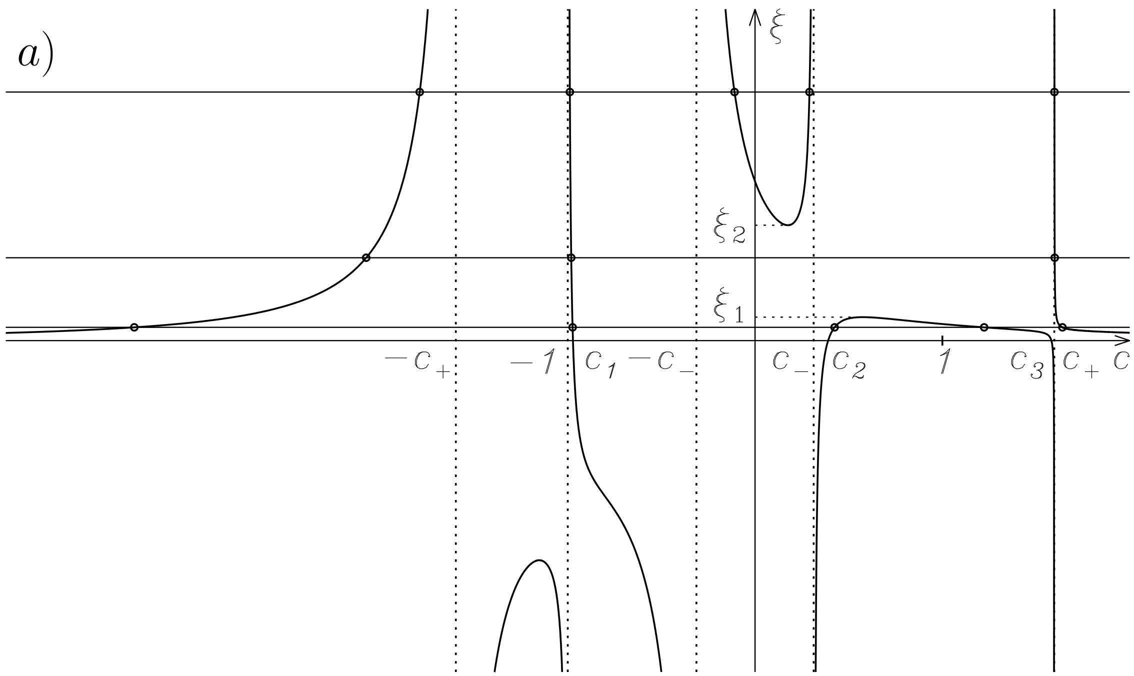
- For any fixed k (2.19) is the fifth-order polynomial equation for ω.
- If all roots of this equation are real, then the perturbation with the fixed k is neutrally stable.
- Complex roots of (2.19) exist in complex conjugate pairs $\Longrightarrow$ if not all roots are real then the perturbation with the fixed k is unstable.

Let us introduce the dimensionless phase velocity $c = \omega/k$. Then (2.19) can be rewritten as

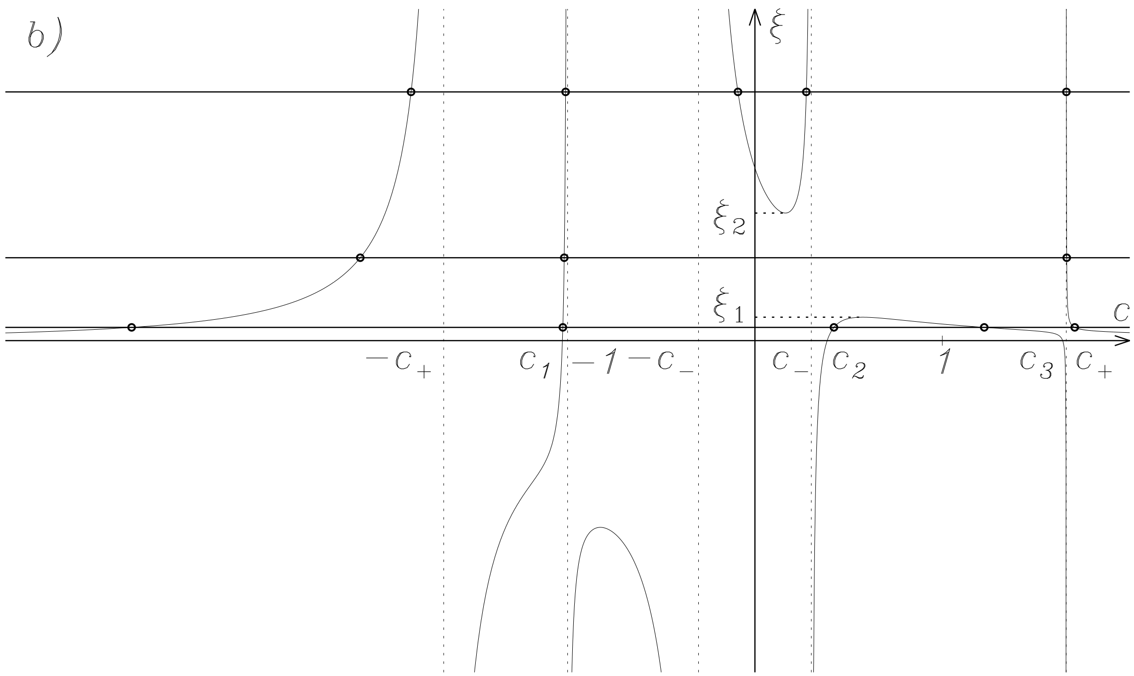
$$k^2 = F(c) \equiv \frac{G(c)}{(c+1)[c^4 - (1 + a^2 + b^2)c^2 + b^2]} \tag{3.1}$$

$$G(c) = 4(c-1)(c^2 - b^2) - a^2(3c - 1) \tag{3.2}$$

The dispersion equation (2.19) has five real roots if and only if the line $\xi = k^2$ in the $c\,\xi$-plane has five intersections with the graph of the function $\xi = F(c)$. The graph of $F(c)$ for $a^2 = 1.4$ and $b = 0.5$ is given below. It can be shown that it remains qualitatively the same for any values of a and b satisfying $a^2 < 2(1 - b^2)$ (case a).



The next figure displays the graph of $F(c)$ for $a^2 = 1.6$ and $b = 0.5$. It can be shown that it remains qualitatively the same for any values of a and b satisfying $a^2 < 2(1 - b^2)$ (case b).



We see that in both cases, a) and b), there are two quantities, ξ_1 and ξ_2, such that there are five intersections of $\xi = k^2$ and $\xi = F(c)$ when either $\xi < \xi_1$ or $\xi > \xi_2$, and only three intersections when $\xi_1 < \xi < \xi_2$.

Hence, we conclude that the circularly polarized Alfvén wave is unstable with respect to normal modes with the wave number k satisfying $\xi_1 < k^2 < \xi_2$.

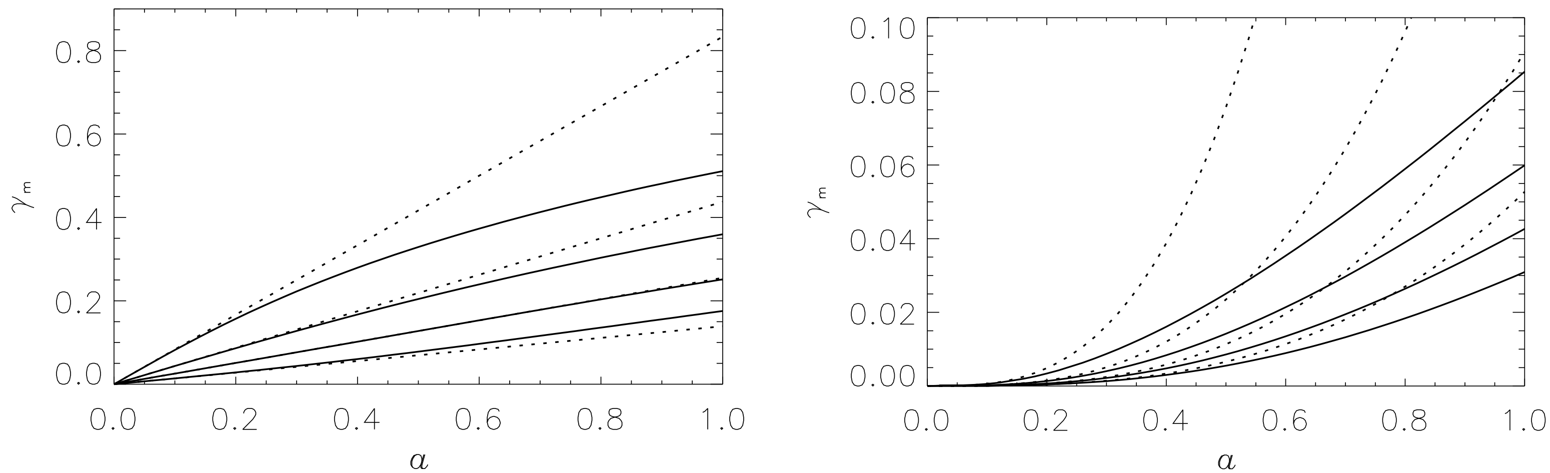
The length of the interval of unstable wave numbers, $\xi_2 - \xi_1$, is a monotonically growing function of a.

When $b < 1$ ($c_S < V_A$) the instability is called the **decay** instability. The reason is that, for $a \ll 1$, the instability is due to decay of the circularly polarized Alfvén wave (also called the **pump wave**) into a forward propagating sound wave and backward propagating Alfvén wave. For $a \ll 1$ and b not very close to 0 and 1 the instability increment, γ_m, is given by

$$\gamma_m = \frac{a\sqrt{1-b}}{2(1+b)\sqrt{b}} \tag{3.3}$$

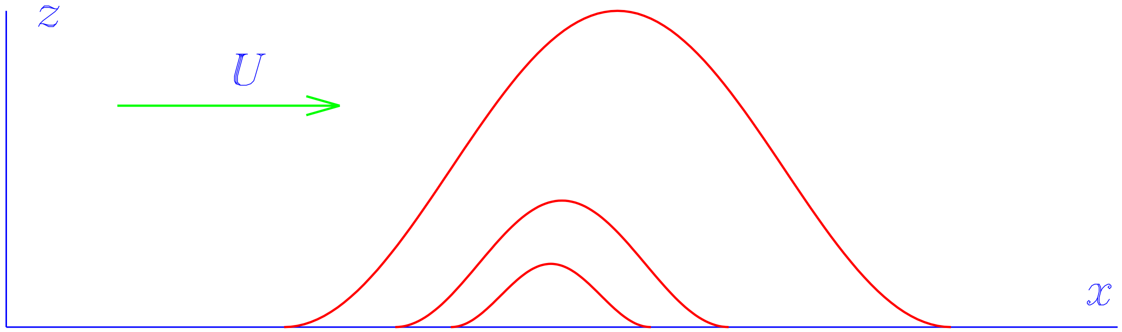
When $b > 1$ ($c_S > V_A$) the instability is called the **beat** instability because, for $a \ll 1$, the instability is due to beating of the pump wave with the sound wave which produces the forward and backward propagating Alfvén waves. For $a \ll 1$ and $b-1 \sim 1$ the instability increment is given by

$$\gamma_m = \frac{a^3}{4\sqrt{2}(b^2-1)^{3/2}} \tag{3.4}$$

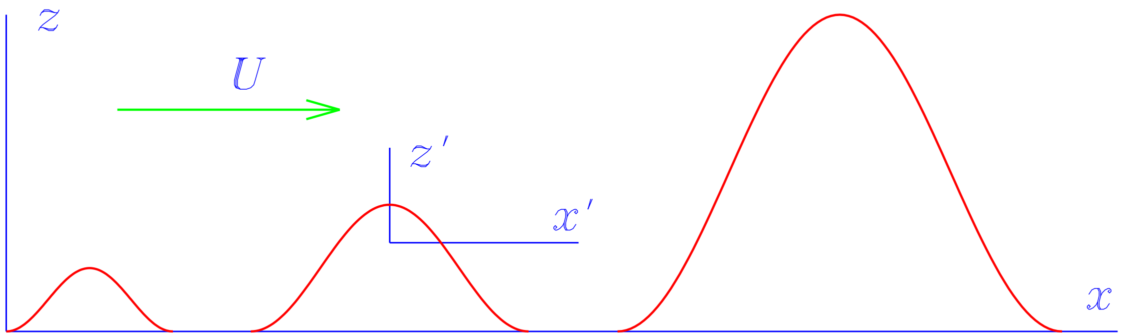


γ_m versus a for the decay (left panel) and beat (right panel) instability. Solid (dashed) lines represent numerical (analytical) results for $b = 0.2$ (top line) in increments of 0.2 up to $b = 0.8$ (bottom line) in the left panel, and for $b = 1.2$ (top line) in increments of 0.2 up to $b = 1.8$ (bottom line) in the right panel.

4. ABSOLUTE AND CONVECTIVE INSTABILITIES: GENERAL CONCEPT



ABSOLUTE INSTABILITY



CONVECTIVE INSTABILITY

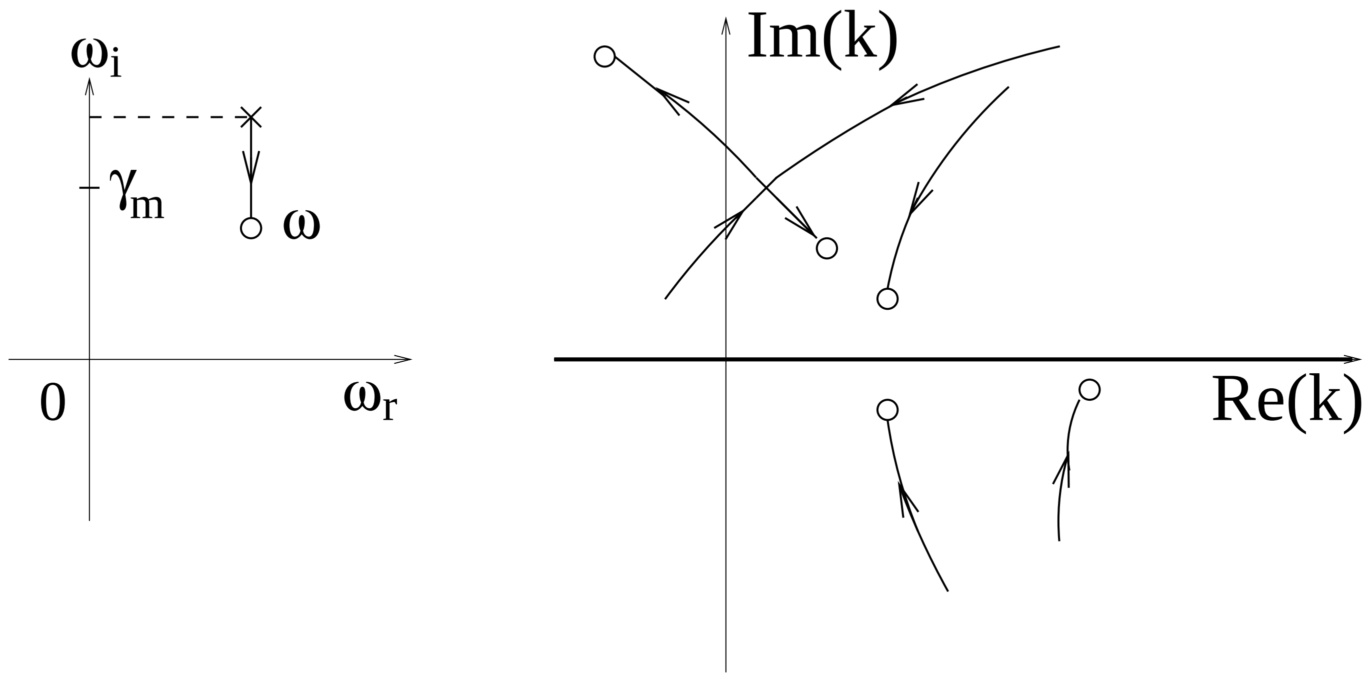
To study absolute and convective instabilities we need to solve the initial value problem for linearized equations, in our case for equations (2.6)–(2.11). To do this we apply the Fourier transform with respect to x and the Laplace transform with respect to t to these equations written in a reference frame moving with velocity $\overline{U}$ with respect to the rest plasma. We find the solution of the transformed equations and then use the inverse Fourier and Laplace transform. As a result we obtain

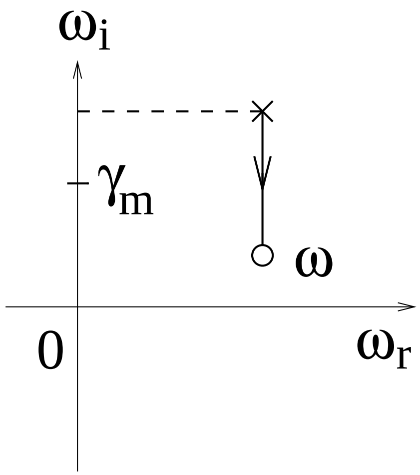
$$\rho'(x,t) = \int_{i\tau-\infty}^{i\tau+\infty} e^{-i\omega t} d\omega \int_{-\infty}^{\infty} \frac{T(k,\omega)}{\widetilde{D}(k,\omega)} e^{ikx} dk \tag{4.1}$$

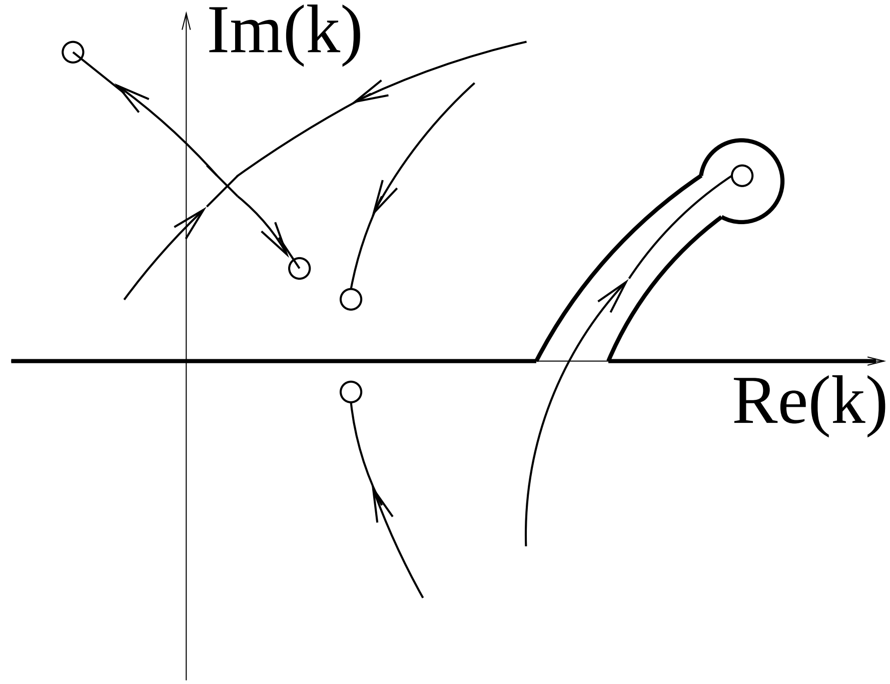
Here $\widetilde{D}(k,\omega) = D(k,\widetilde{\omega})$, where $D(k,\omega) = 0$ is the dispersion equation given by (2.19); $\widetilde{\omega} = \omega + kU$ is the Doppler-shifted frequency and $U = \overline{U}/v_A$. $T(k,\omega)$ is determined by the initial conditions and is not important for what follows. $\Im(\omega) = \tau$ is the Bromwich integration contour, $\tau > \gamma_m$. The other variables are given by similar expressions.

To distinct between the absolute and convective instabilities we need to study the asymptotic behaviour of $\rho'(x,t)$ as $t \to \infty$.

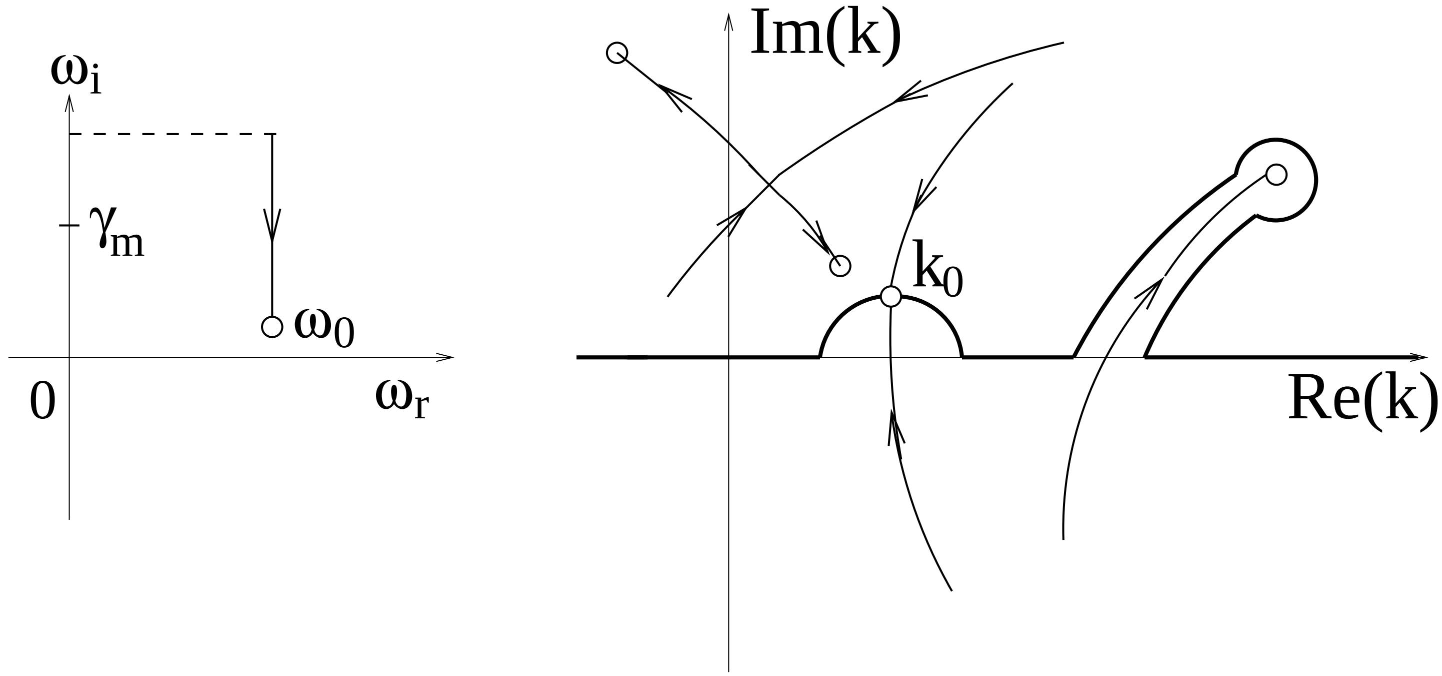
We are moving the Bromwich integration contour downward point by point. Let us write $\omega = \omega_r + i\omega_i$, fix ω_r and start to decrease ω_i.





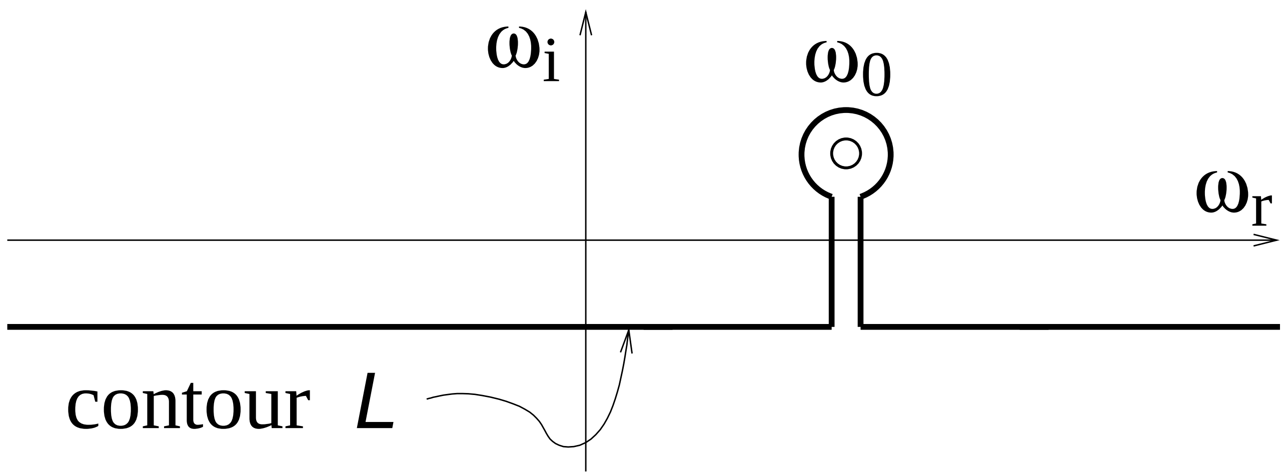


We can band the integration contour in the complex k-plane because the integrand in (4.1) is an analytic function of k.



The two roots that came from different sides of the contour "**pinch**" the contour. The double root k_0 that they form is called the **pinching** root. The value of ω_i corresponding to pinching depends on ω_r. Let pinching occurs first time (i.e. for the largest value of ω_i) just when $\omega_r = \omega_{0r}$. Then all point of integration contour in the complex ω-plane can be moved below ω_0, so that ω_0 is the apex point of the deformed contour.

To avoid singularity we have to stop moving the contour near ω_0 slightly before the pinching occurs. As a result we obtain the following integration contour in the complex ω-plain:



The sketch of integration contour in the complex ω-plane.

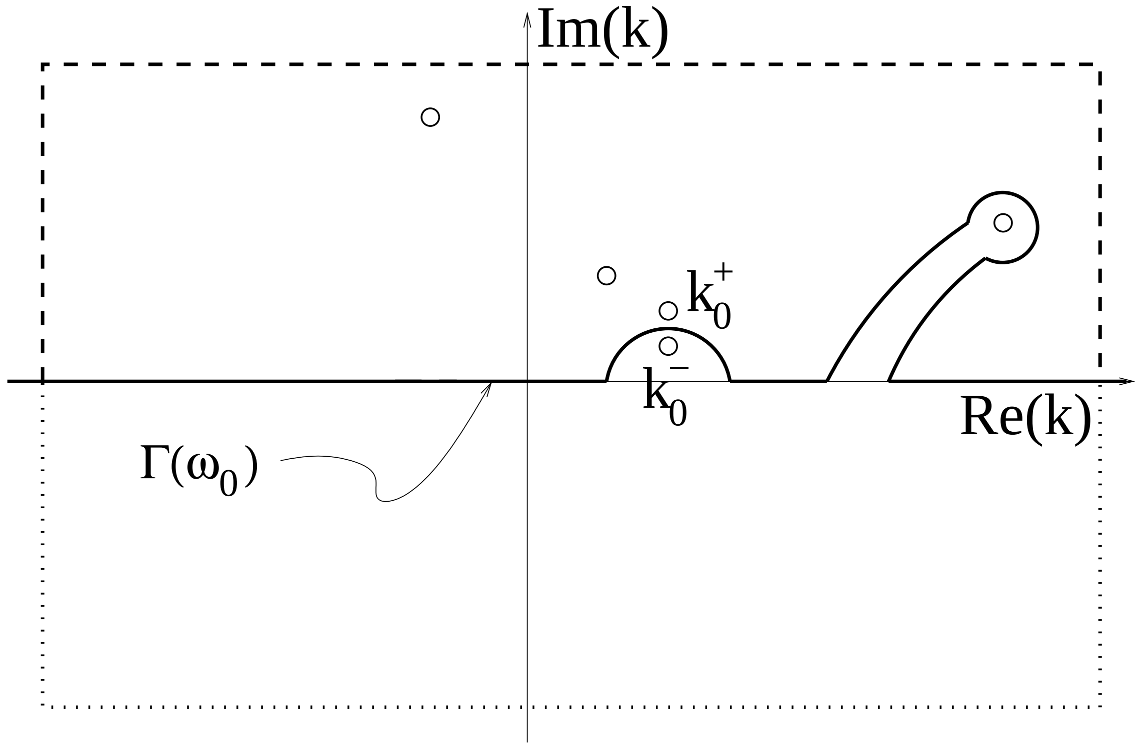
Expression (4.1) transforms to

$$\rho'(x,t) = \int_{\mathcal{L}} e^{-i\omega t} d\omega \int_{\Gamma(\omega)} \frac{T(k,\omega)}{\widetilde{D}(k,\omega)} e^{ikx} dk \tag{4.2}$$

We introduce the notation

$$F(\omega, x) = \int_{\Gamma(\omega)} \frac{T(k,\omega)}{\widetilde{D}(k,\omega)} e^{ikx} dk \tag{4.3}$$

In small vicinity of ω_0 we can take $\Gamma(\omega) = \Gamma(\omega_0)$. To calculate $F(\omega, x)$ we use the residue theorem.



The dashed (dotted) contour is used for closure when $x > 0$ ($x < 0$).

As a result we obtain

$$
\begin{aligned}
F(\omega, x) &= 2\pi i \,\mathrm{sgn}(x) \sum_n \operatorname*{res}_{k=k_n(\omega)} \left(\frac{T(k,\omega)}{\widetilde{D}(k,\omega)} \, e^{ikx} \right) \\
&= 2\pi i \,\mathrm{sgn}(x) \sum_n \frac{T(k_n(\omega),\omega)}{\left.\dfrac{\partial \widetilde{D}}{\partial k}\right|_{k=k_n(\omega)}} \, e^{ixk_n(\omega)}
\end{aligned}
\tag{4.4}
$$

The sum is taken over all zeros of $\widetilde{D}(k,\omega)$ considered as function of k in the upper (lower) part of complex k-plane when $x > 0$ ($x < 0$).

k_0 is double root of $\widetilde{D}(k,\omega_0) \Longrightarrow \left.\dfrac{\partial \widetilde{D}}{\partial k}\right|_{\substack{k=k_0 \\ \omega=\omega_0}} = 0 \Longrightarrow$ Taylor expansion of $\widetilde{D}(k,\omega)$ near (k_0,ω_0) takes the form

$$
\widetilde{D}(k,\omega) = \left(\frac{\partial \widetilde{D}}{\partial \omega} \right)_0 (\omega - \omega_0) + \frac{1}{2} \left(\frac{\partial^2 \widetilde{D}}{\partial k^2} \right)_0 (k - k_0)^2 + \ldots \tag{4.5}
$$

Let us denote as $k_0^+(\omega)$ and $k_0^-(\omega)$ the two roots that collide and form the double root k_0, so that $k_0^\pm(\omega_0) = k_0$, $\Im(k_0^+(\omega)) > 0$, $\Im(k_0^-(\omega)) < 0$ when $\Im(\omega) > \gamma_m$. Then (4.5) $\Longrightarrow$

$$k_0^\pm(\omega) \approx k_0 \pm \sqrt{g(\omega - \omega_0)}, \quad g = -2\left(\frac{\partial \widetilde{D}}{\partial \omega}\right)_0 \Big/ \left(\frac{\partial^2 \widetilde{D}}{\partial k^2}\right)_0 \tag{4.6}$$

in a vicinity of ω_0. Using (4.6) we obtain

$$\left.\frac{\partial \widetilde{D}}{\partial k}\right|_{k=k_0^\pm} \approx \left(\frac{\partial^2 \widetilde{D}}{\partial k^2}\right)_0 (k - k_0) = \pm\left(\frac{\partial^2 \widetilde{D}}{\partial k^2}\right)_0 \sqrt{g(\omega - \omega_0)} \tag{4.7}$$

We see that only term with $k_n = k_0^+$ ($k_n = k_0^-$) is singular at $\omega = \omega_0$ in (4.4) $\Longrightarrow$ in small vicinity of ω_0 this term strongly dominates other terms $\Longrightarrow$ in this vicinity

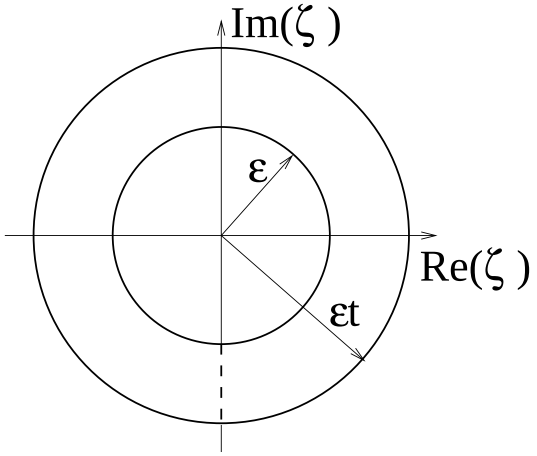
$$F(\omega, x) \approx h(\omega - \omega_0)^{-1/2} e^{ik_0 x}, \quad h = \frac{2\pi i T(k_0, \omega_0)}{\left.\frac{\partial^2 \widetilde{D}}{\partial k^2}\right|_0 \sqrt{g}} \tag{4.8}$$

For large t the main contribution in the expression (4.2) for $\rho'(x,t)$ comes from the integral over the small circle with the centre at ω_0. On this circle we can use the approximate expression (4.8) $\Longrightarrow$ for $t \to \infty$ we obtain

$$\rho'(x,t) = he^{ik_0x} \int_{\mathcal{C}} (\omega - \omega_0)^{-1/2} e^{-i\omega t} d\omega \tag{4.9}$$

where $\mathcal{C}$ is the circle of radius ε with centre at ω_0. We make the variable substitution $\omega = \omega_0 + \zeta/t$. Then

$$\int_{\mathcal{C}} (\omega - \omega_0)^{-1/2} e^{-i\omega t} d\omega = t^{-1/2} e^{-i\omega_0 t} \int_{\mathcal{C}_t} \zeta^{-1/2} e^{-i\zeta} d\zeta \tag{4.10}$$



where $\mathcal{C}_t$ is the circle of radius εt with the centre at the coordinate origin.

The figure together with the residue theorem show that we can use the integration contour $\mathcal{C}_1$ instead of $\mathcal{C}_t$, where $\mathcal{C}_1$ has the radius ε.

Equations (4.9) and (4.10) show that, as $t \to \infty$,

$$\rho'(x,t) = \text{const} \times t^{-1/2} \exp[i(k_0 x - \omega_0 t)] \tag{4.11}$$

Hence, the instability is absolute if $\Im(\omega_0) > 0$, and convective if $\Im(\omega_0) \leq 0$.

In accordance with our analysis we can divide studying the absolute and convective instabilities in the following **five steps**:

(i) Derive the dispersion equation using the normal mode analysis, and calculate the maximum growth rate of the instability γ_M.

(ii) Calculate all double-roots of the dispersion equation, which are determined by the system of two equations:

$$\widetilde{D}(k,\omega) = 0, \qquad \frac{\partial \widetilde{D}}{\partial k} = 0 \tag{4.12}$$

(4.12) determines double roots, k, and corresponding values of ω.

(iii) Retain only those pairs of solutions to (4.12), (k, ω), that satisfying the inequality

$$0 < \omega_i \leq \gamma_m. \tag{4.13}$$

If there are no such pairs, then the instability is **convective**.

(iv) Among all pairs (k, ω) satisfying (4.13) retain only those where k is a pinching double root. To verify that k is pinching, we fix $\Re(\omega)$ and increase the $\Im(\omega)$ from ω_i to $\gamma_M + \epsilon$. We obtain the trajectories of the two k-roots that collide at the point k of the complex k-plane. If the end-points of these trajectories are on different sides of the real axis in the complex k-plane, then the double root is pinching. Otherwise it is non-pinching.

(v) Finally, among all pairs (k, ω) satisfying (4.13) and the condition that k is pinching, we choose one with the largest ω_i (of course it is possible that there are a few pairs with the same ω_i). Using the notation (k_0, ω_0) for this pair, we obtain that the asymptotic behaviour of the density perturbation is given by (4.11).

5. ABSOLUTE AND CONVECTIVE INSTABILITIES: APPLICATION TO CIRCULARLY POLARIZED ALFVÉN WAVES

5.1. Decay instability ($c_S < V_A$)

Previously we used c to denote the phase velocity. Now we use it to denote Doppler-shifter phase velocity. So, in what follows, $c = \widetilde{\omega}/k$. Now equation $\widetilde{D}(k,\omega) = 0$ takes the form given by (3.1) and (3.2), and the system of two equations, (4.12), can be written as

$$k^2 = \frac{4(c-1)(c^2-b^2) - a^2(3c-1)}{(c+1)[c^4 - (1+a^2+b^2)c^2 + b^2]} \tag{5.1}$$

$$\begin{aligned} &4(1+U)(c+1)(c-1)^2(c^2-b^2)^2 - a^2\left\{\left[c^6 + 4c^5 - 3c^4 - 2(1+3b^2)c^3\right.\right.\\ &\quad\left. + 3b^2c^2 + 4b^2c - b^2\right] + U\left[6c^5 - 2c^4 - (5+7b^2)c^3 + 4b^2c^2\right.\\ &\quad\left.\left. + (1+5b^2)c - 2b^2\right]\right\} + a^4\left[2c^3 + U(3c^3 - c)\right] = 0 \end{aligned} \tag{5.2}$$

Equation (5.2) is the 7th order algebraic equation $\Longrightarrow$ it has 7 roots. For each root of (5.2) we will find two values of k from (5.1). For each k we obtain corresponding value of ω using $\omega = (c - U)k$. Hence, we have 14 pairs (k, ω), where k is the double root of the dispersion equation, and ω is the corresponding value of frequency.

To make analytical progress we assume $a \ll 1$. Then it is possible to calculate all 14 pairs of (k, ω) and show that only 6 satisfy the condition (4.13). It is also possible to show that, among these 6 roots only two are **pinching** when $-1 < U < b$, and none root is pinching otherwise. The pinching roots are given by

$$k_{\pm} = \pm\frac{2}{1+b} + \frac{ia(1-b)^{1/2}(1-b+2U)}{2(1+b)^2[b(U+1)(b-U)]^{1/2}} + \mathcal{O}(a^2) \qquad (5.3)$$

The corresponding values of ω are given by

$$\omega_\pm = \pm\frac{2(b-U)}{1+b} + \frac{ia\left[(1-b)(U+1)(b-U)\right]^{1/2}}{b^{1/2}(1+b)^2} + \mathcal{O}(a^2) \qquad (5.4)$$

We see that $\Im(\omega_+) = \Im(\omega_-) > 0$ when $-1 < U < b$. Since there are no pinching roots when either $U < -1$, or $U > b$, we conclude that the instability is **absolute** when

$$\boxed{-1 < U < b} \qquad (5.5)$$

and **convective** otherwise.

$$\Im(\omega_\pm) = \gamma_m \quad \text{for} \quad U = \frac{b-1}{2}$$

5.2. Beat instability ($c_S > V_A$)

We solve the same system (5.1)–(5.2) and once again obtain 14 pairs of (k, ω). Once again only 6 pairs satisfy (4.13). Now there are two pinching roots when $-1 < U < 1$, and no pinching roots otherwise. The pinching roots are given by

$$k_{\pm} = \pm \left\{ 1 + \frac{a^2}{4(b^2 - 1)} \right\} + \frac{ia^3 U}{4(b^2 - 1)^{3/2}[2(1 - U^2)]^{1/2}} + \mathcal{O}(a^4) \quad (5.6)$$

The corresponding values of ω are given by

$$\omega_{\pm} = \pm \left\{ 1 - U - \frac{a^2(1 + U)}{4(b^2 - 1)} \right\} + \frac{ia^3}{8} \left[\frac{2(1 - U^2)}{(b^2 - 1)^3} \right]^{1/2} + \mathcal{O}(a^4) \quad (5.7)$$

We see that $\Im(\omega_+) = \Im(\omega_-) > 0$ when $-1 < U < 1$.

Since there are no pinching roots when either $U < -1$, or $U > 1$, we conclude that the instability is **absolute** when

$$\boxed{-1 < U < 1} \tag{5.8}$$

and **convective** otherwise.

$$\Im(\omega_{\pm}) = \gamma_m \text{ for } U = 0$$

6. APPLICATION TO SOLAR WIND

The solar wind speed is ~ 500 km/s, $V_A \sim 50$ km/s, the speed of any space station in the solar reference frame is $\ll 500$ km/s $\Longrightarrow$ $\overline{U} \sim 500$ km/s is the rest plasma reference frame $\Longrightarrow$ $|U| \sim 10$ $\Longrightarrow$ $|U| \gg b,\ 1$ $\Longrightarrow$ the instability observed by any space station is **convective** both when $b < 1$ and when $b > 1$.

Consider a circularly polarized Alfvén wave propagating in the anti-solar direction. Let its amplitude be $a = 0.2$, and its period in the solar wind reference frame 3 hours, which corresponds to $\omega_0 \approx 5.8 \times 10^{-4}$ s^{-1}. Assume that this wave is perturbed at a distance $\ll 1\ a.u.$ This perturbation will be convected from the sun by the solar wind.

The time interval after which the perturbation arrives at the Earth orbit is $t_{\rm travel} \approx 1\ a.u./(500\ \mathrm{km/s}) \approx 3 \times 10^5$ s. The dimensional increment of the perturbation is $\omega_0 \gamma_m \Longrightarrow$ its amplitude will increase by

$$\exp(\omega_0 \gamma_m t_{\rm travel})$$

before it reaches the Earth orbit.

Consider first the case when $c_S < V_A$ $(b < 1)$, so the instability is **decay**. In that case

$$\omega_0 \gamma_m t_{\rm travel} \approx \frac{a\sqrt{1-b}}{2(1+b)\sqrt{b}}\, \omega_0 t_{\rm travel} \approx 12$$

for $b \sim 0.5$, so that $\exp(\omega_0 \gamma_m t_{\rm travel}) \approx 1.6 \times 10^5 \Longrightarrow$ nonlinear regime.

When $c_S > V_A$ $(b > 1)$, **beat** instability,

$$\omega_0 \gamma_m t_{\rm travel} \approx \frac{a^3}{4\sqrt{2}(b^2-1)^{3/2}}\, \omega_0 t_{\rm travel} \approx 0.05$$

for $b \sim 2$, so that $\exp(\omega_0 \gamma_m t_{\rm travel}) \approx 1.05$.